INTEGRAL IDENTITIES INVOLVING ZONAL POLYNOMIALS

Gordon M. Kaufman

July, 1965

136-65

INTEGRAL IDENTITIES INVOLVING ZONAL POLYNOMIALS

Gordon M. Kaufman

July, 1965

136-65

This work was supported in part by Grant No. GS-95 from the National Science Foundation.

INTEGRAL IDENTITIES INVOLVING ZONAL POLYNOMIALS

Gordon M. Kaufman

1. Introduction

Constantine [3] and James [5] have treated a wide variety of distribution problems in multivariate analysis using hypergeometric functions ${}_pF_q$ of matrix argument and their expansions in zonal polynomials. The great advantage of these two notions, defined in section 1 below, is that they unify the mathematical characterizations of a wide class of multivariate density functions, functions that otherwise would have to be expressed in inordinately difficult multiple series; e.g. the non-central Wishart density, the non-central multivariate F density, the density of canonical correlations in the non-null case, etc.

Almost all of their results rest on three major properties of zonal polynomials (See James [5] J(22), J(23), J(24)). The main purpose of this paper is to record some additional properties of zonal polynomials that are needed in order to do a Bayesian analysis of certain multivariate data generating processes closely related to the multivariate Normal process. In particular, we show how some of the integral identities shown here can be used (a) in the analysis of the econometrician's simultaneous equations system from a Bayesian point of view; (b) to give an explicit series representation for the characteristic function of the generalized inverted beta density defined in section 5; (c) to prove an analogue of one of Constantine's major theorem's quoted in section 2 below; (d) to provide a further extension of an integral identity due essentially to Bellman [2] (See Olkin [8] also) that is extremely useful in building Bayesian extensions of natural conjugate families for the multivariate Normal process as reported in Ando and Kaufman [1].

To foreshadow the applications that flow from some of the results of this paper and to illustrate the kind of problem that generates interest in them, suppose we wish to analyze the following set of stochastic equations Bayesianly:

$$\underline{\underline{B}}\ \underline{\tilde{y}}^{(j)} + \underline{\underline{\Gamma}}\ \underline{z}^{(j)} = \underline{\tilde{u}}^{(j)} \qquad , \qquad j=1,2,\ldots \tag{1.1}$$

where $\underline{\underline{B}}$ and $\underline{\underline{\Gamma}}$ are (m x m) and (m x r) coefficient matrices, fixed for all j, $\underline{z}^{(j)}$ is an (r x 1) vector of predetermined variables and $\underline{\tilde{y}}^{(j)}$ and $\underline{\tilde{u}}^{(j)}$ are (m x 1) and (r x 1) random vectors respectively. We assume that $\{\underline{\tilde{u}}^{(j)}, j=1,2,\ldots\}$ is a sequence of mutually independent, identically Normal random vectors with mean $\underline{0}$ and PDS covariance matrix $\underline{\underline{\Sigma}} \equiv \underline{\underline{h}}^{-1}$; and $\underline{\underline{B}}$ is non-singular. One observes $(\underline{y}^{(j)}, \underline{z}^{(j)})$ j=1,2,... but neither $\underline{\underline{B}}$, nor $\underline{\underline{\Gamma}}$ nor $\underline{\underline{h}}$ is known with certainty. As Bayesians we wish to regard $\underline{\underline{\tilde{B}}}$, $\underline{\underline{\tilde{\Gamma}}}$, and $\underline{\underline{\tilde{h}}}$ as jointly distributed random variables, place a joint prior density on $(\underline{\underline{\tilde{B}}}, \underline{\underline{\tilde{\Gamma}}}, \underline{\underline{\tilde{h}}})$ and then do a variety of calculations. Of particular interest to econometricians is the joint density of $(\underline{\underline{\tilde{\Gamma}}}, \underline{\underline{\tilde{h}}})$ unconditional as regards $\underline{\underline{\tilde{B}}}$, the marginal density of $\underline{\underline{\tilde{\Gamma}}}$, and the particulars of blending the prior on $(\underline{\underline{\tilde{\Gamma}}}, \underline{\underline{\tilde{h}}}, \underline{\underline{\tilde{B}}})$ with objective sample evidence $\{(\underline{y}^{(j)}, \underline{z}^{(j)}), j=1,2,\ldots,n\}$ via Bayes Theorem to find the posterior joint density of $(\underline{\underline{\tilde{\Gamma}}}, \underline{\underline{\tilde{h}}}, \underline{\underline{\tilde{B}}})$. Here we show how to calculate the first mentioned densities when the prior on $(\underline{\underline{\tilde{\Gamma}}}, \underline{\underline{\tilde{h}}})$ given $\underline{\underline{\tilde{B}}} = \underline{\underline{B}}$ is in the natural conjugate family† of priors (Normal-Wishart) as defined in [1] and a prior with kernel

$$e^{-\frac{1}{2}\mathrm{tr}\ \underline{\underline{\psi}}[\underline{\underline{B}}-\underline{\underline{\bar{B}}}]\ \underline{\underline{E}}[\underline{\underline{B}}-\underline{\underline{\bar{B}}}]^t}\ |\underline{\underline{B}}\ \underline{\underline{B}}^t|^{\alpha} \quad , \quad \underline{\underline{\psi}},\ \underline{\underline{E}} > \underline{\underline{0}},\ \alpha > 0 \quad , \tag{1.2}$$

---

†See Ref. [9] for a detailed discussion of the notion of natural conjugate priors.

is assigned to $\tilde{\underline{\underline{B}}}$ with range set $M_{m,m}$. This is a "natural" family of normed priors to assign to $\tilde{\underline{\underline{B}}}$. We leave the justification for assigning this class to $\tilde{\underline{\underline{B}}}$ and the details of prior-posterior analysis under the above assignments to another paper that will deal more fully with (1.1), but show that the joint density of $(\tilde{\underline{\underline{\Gamma}}}, \tilde{\underline{\underline{h}}})$ unconditional as regards $\tilde{\underline{\underline{B}}}$ is expressible as a product of a Wishart density and a hypergeometric function ${}_1F_1$ of matrix argument, while the marginal of $\tilde{\underline{\underline{\Gamma}}}$ is essentially ${}_2F_1$ with argument a complicated matric function of $\underline{\underline{\Gamma}}$. The function ${}_1F_1$ is a generalization of the classical Laguerre polynomials, and Herz [4] shows that it is computable in terms of classical Laguerre polynomials of single argument. The function ${}_2F_1$ is a generalization of the Gaussian hypergeometric function.

1.1 Notation

Throughout all matrices $\underline{\underline{Z}}$ are understood to be (m x m) unless otherwise stated. We denote the set of all (m x m) non-singular matrices by $M_{m,m}$ and the cone generated by the set of all (m x m) positive definite symmetric matrices by $\underline{\underline{h}} > \underline{\underline{0}}$. A tilde denotes a random variable; e.g. $\tilde{\underline{\underline{B}}}$. And Re $\underline{\underline{T}}$ and tr $\underline{\underline{T}}$ denote the real part of $\underline{\underline{T}}$ when $\underline{\underline{T}}$ is complex and the trace of $\underline{\underline{T}}$ respectively. We shall have occasion to integrate over the real orthogonal group of order m (the set of all orthogonal matrices of order m) and label it O(m).

By $J(\underline{\underline{B}} \to \underline{\underline{W}})$ we mean the Jacobian of the matrix transform from $\underline{\underline{B}}$ to $\underline{\underline{W}}$.

The unique PDS square root of a PDS matrix $\underline{\underline{R}}$ is written $\underline{\underline{R}}^{\frac{1}{2}}$ and $|\underline{\underline{R}}|$ denotes the determinant of $\underline{\underline{R}}$. We also need the symbols

$$\underline{\underline{h}}^{[j]} = \begin{bmatrix} h_{11} & \cdots & h_{ij} \\ \vdots & & \vdots \\ h_{j1} & \cdots & h_{jj} \end{bmatrix} \quad \text{and} \quad \underline{\underline{h}}_{[j]} = \begin{bmatrix} h_{j+1,j+1} & \cdots & h_{j+1,m} \\ \vdots & & \vdots \\ h_{m,j+1} & \cdots & h_{m,m} \end{bmatrix}.$$

The symbol $\gamma = \frac{1}{2}(m+1)$ throughout. Also $\Gamma_m(x) = \pi^{\frac{1}{4}m(m-1)} \prod_{i=1}^{m} \Gamma(x-\frac{1}{2}(i-1))$, and for a partition $\kappa = (k_1,\ldots,k_m)$ of the positive integer k into non-negative integers $k_1 \geq k_2 \geq \ldots \geq k_m$, $\Gamma_m(x, \kappa) = \pi^{\frac{1}{4}m(m-1)} \prod_{i=1}^{m} \Gamma(x + k_i - \frac{1}{2}(i-1))$. We remark that if, as is conventional, we define $(x)_o = 1$, $(x)_n = x(x+1)\ldots(x+n-1)$ and for a partition κ, $(x)_\kappa = \prod_{i=1}^{m} (x - \frac{1}{2}(i-1))_{k_i}$, then $(x)_\kappa = \Gamma_m(x,\kappa)/\Gamma_m(x)$.

A formulae labelled with a letter refers to the same numbered formula in the article by the author whose initial the letter is (see bibliography); e.g. C(11) denotes formula (11) in Constantine [3]. Formulae prelabelled with an I are the main identities proven here; e.g. I(1) through I(6).

For notational compactness, whenever the argument of a symmetric function appears as the sum of an arbitrary (m x m) matrix, say $\underline{\underline{Z}}$, and its transpose, we shall write the argument as $2\underline{\underline{Z}}$ rather than $\underline{\underline{Z}} + \underline{\underline{Z}}^t$; e.g. $\mathrm{tr}(\underline{\underline{Z}} + \underline{\underline{Z}}^t) = \mathrm{tr}\, 2\, \underline{\underline{Z}}$.

2. Zonal Polynomials and Hypergeometric Functions of Matrix Argument

Constantine [3] and James [5], discuss zonal polynomials in detail. Here we give their definition, quoting Constantine, and state those of their properties needed in the sequel. Constantine [3] gives their definition as follows:

"Let $\underline{\underline{S}}$ be a positive definite, symmetric m x m matrix, and $\psi(\underline{\underline{S}})$ a polynomial in the elements of $\underline{\underline{S}}$. Then, the transformation

$$\psi(\underline{\underline{S}}) \rightarrow \psi(\underline{\underline{L}}^{-1}\ \underline{\underline{S}}\ \underline{\underline{L}}'^{-1}) \quad , \quad \underline{\underline{L}} \in GL(m) \quad , \tag{2.1}$$

defines a representation of the real linear group GL(m) in the vector space of all polynomials in $\underline{\underline{S}}$. The space V_k of homogeneous polynomials of degree k is invariant under the transformations (2.1) and decomposes into the direct sum of irreducible subspaces $V_k = \Sigma_k \oplus V_{k,\kappa}$ where $\kappa = (k_1, k_2, \ldots, k_m)$, $k_1 \geq k_2 \geq \ldots \geq k_m \geq 0$, runs over all partitions of k into not more than m parts. Each $V_{k,\kappa}$ contains a unique one dimensional subspace invariant under the orthogonal group O(m). These subspaces are generated by the zonal polynomials, $C_\kappa(\underline{\underline{S}})$. Being invariant under the orthogonal group, i.e.,

$$C_\kappa(\underline{\underline{H}}'\ \underline{\underline{S}}\ \underline{\underline{H}}) = C_\kappa(\underline{\underline{S}}) \quad , \quad \underline{\underline{H}} \in O(m) \tag{2.2}$$

they are homogeneous symmetric polynomials in the characteristic roots of $\underline{\underline{S}}$.

The zonal polynomials were defined above only for positive definite symmetric matrices $\underline{\underline{S}}$. However, since they are polynomials in the characteristic roots of $\underline{\underline{S}}$, their definition may be extended to arbitrary complex symmetric matrices. Furthermore, if $\underline{\underline{S}}$ is a symmetric matrix, and $\underline{\underline{R}}$ is a positive definite symmetric matrix, then the roots of $\underline{\underline{R}}\ \underline{\underline{S}}$ are the same as those of $\underline{\underline{R}}^{\frac{1}{2}}\ \underline{\underline{S}}\ \underline{\underline{R}}^{\frac{1}{2}}$ where $\underline{\underline{R}}^{\frac{1}{2}}$ is the (unique) positive definite square root of $\underline{\underline{R}}$. Hence, one may define $C_\kappa(\underline{\underline{R}}\ \underline{\underline{S}}) = C_\kappa(\underline{\underline{R}}^{\frac{1}{2}}\ \underline{\underline{S}}\ \underline{\underline{R}}^{\frac{1}{2}})$.

The fundamental property of the zonal polynomials is given by the following integral, proved in [6]:

$$\int_{O(m)} C_\kappa(\underline{\underline{H}}'\ \underline{\underline{S}}\ \underline{\underline{H}}\ \underline{\underline{T}})\ d(\underline{\underline{H}}) = C_\kappa(\underline{\underline{S}})\ C_\kappa(\underline{\underline{T}})/C_\kappa(\underline{\underline{I}}) \quad , \qquad C(6)$$

where $\underline{\underline{I}}$ is the identity matrix, and $d(\underline{\underline{H}})$ is the invariant Haar measure on the orthogonal group, normalized to make the volume of the group manifold unity."

A property of interest here is for arbitrary (r x m) $\underline{\underline{X}}$,

$$\int_{O(m)} (\mathrm{tr}\ \underline{\underline{X}}\ \underline{\underline{Q}})^{2k}\, d\underline{\underline{Q}} = \sum_{\kappa} \frac{(\frac{1}{2})_k}{(\frac{1}{2}m)_\kappa}\ C_\kappa(\underline{\underline{X}}\ \underline{\underline{X}}^t) \quad . \qquad J(22)$$

In order to prove Theorem 2 below, we need two additional properties of zonal polynomials established by Constantine. Order the partitions of k lexicographically; e.g. if $\kappa = (k_1,k_2,\ldots,k_m)$ and $\tau = (t_1,t_2,\ldots,t_m)$ are two partitions of k, then define $k > \tau$ if $k_1 = t_1,\ldots,k_i = t_i$, $k_{i+1} > t_{i+1}$. Let $s_1,\ldots,s_m$ denote the characteristic roots of the PDS matrix $\underline{\underline{S}}$. Constantine [3] shows that if monomials $s_1^{n_1}, s_2^{n_2},\ldots,s_m^{n_m}$ appearing in the expansion of $C_\kappa(\underline{\underline{S}})$ into a sum of such polynomials are ordered lexicographically, the term of "highest weight" occurring is $s_1^{k_1},\ldots,s_m^{k_m}$ and

$$C_\kappa(\underline{\underline{S}}) = d_{\kappa,\kappa}\ s_1^{k_1},\ldots,s_m^{k_m} + \text{"lower terms"} \quad . \qquad (2.3)$$

Here $d_{\kappa,\kappa}$ is a constant resulting from the inversion of an expression of a group character as a linear combination of zonal polynomials. It will cancel when (2.3) is in use here and need not be explicitly defined.

Two additional properties of primary interest to us are (Constantine [3] (11))

$$C_\kappa(\underline{\underline{S}}) = d_{\kappa,\kappa}\ |\underline{\underline{S}}^{[1]}|^{k_1-k_2}\ |\underline{\underline{S}}^{[2]}|^{k_2-k_3} \cdots |\underline{\underline{S}}^{[m]}|^{k_m} + \text{"lower terms"} \quad C(11)$$

and if $\underline{\underline{T}}$ is diagonal, with diagonal elements $t_1,t_2,\ldots,t_m$

$$C_\kappa(\underline{\underline{S}}\ \underline{\underline{T}}) = d_{\kappa,\kappa}\, t_1^{k_1} \cdots t_m^{k_m} \, |\underline{\underline{S}}^{[1]}|^{k_1-k_2} \, |\underline{\underline{S}}^{[2]}|^{k_2-k_3} \cdots |\underline{\underline{S}}^{[m]}|^{k_m} + \ldots \qquad C(11)$$

The hypergeometric functions of matrix argument that we mentioned at the outset are defined (see [3] and [5]) in terms of zonal polynomials like this:

Definition: For integers p and q

$$_pF_q(a_1,\ldots,a_p;\ b_1,\ldots,b_q;\ \underline{\underline{S}}) \equiv \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_\kappa \cdots (a_p)_\kappa \, C_\kappa(\underline{\underline{S}})}{(b_1)_\kappa \cdots (b_q)_\kappa \, k!} \qquad J(10)$$

where $a_1,\ldots,a_p$, $b_1,\ldots,b_q$ are real of complex constants and for any given partition $\kappa = (k_1,\ldots,k_m)$, $(a)_\kappa \equiv \prod_{i=1}^{m} (a-\tfrac{1}{2}(i-1))_{k_i}$ and $(a)_{k_i} = a(a+1) \ldots (a+k_i-1)$.

Hypergeometric functions of two arguments are defined analogously:

Definition:

$$_pF_q(a_1,\ldots,a_p;\ b_1,\ldots,b_q;\ \underline{\underline{S}},\ \underline{\underline{T}}) \equiv \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_\kappa \cdots (a_p)_\kappa \, C_\kappa(\underline{\underline{S}})\, C_\kappa(\underline{\underline{T}})}{(b_1)_\kappa \cdots (b_q)_\kappa \, k!\, C_\kappa(\underline{\underline{I}})} \qquad J(13)$$

Herz [4] defined the above system of hypergeometric functions of matrix argument in terms of Laplace and inverse Laplace transform and used them to extend many classical univariate formulae in an elegant fashion:

$$_{p+1}F_q(a_1,\ldots,a_p,\ \alpha;\ b_1,\ldots,b_q;\ -\underline{\underline{Z}}^{-1})\ |\underline{\underline{Z}}|^{-\alpha}$$

$$= \frac{1}{\Gamma_m(\alpha)} \int_{\underline{\underline{h}} > \underline{\underline{0}}} e^{-\mathrm{tr}\ \underline{\underline{h}}\ \underline{\underline{Z}}}\ {}_pF_q(\alpha_1,\ldots,\alpha_p;\ \beta_1,\ldots,\beta_q;\ -\underline{\underline{h}})\ |\underline{\underline{h}}|^{\alpha-\gamma} d\underline{\underline{h}}\ . \qquad H(2.1)$$

and

$$_pF_{q+1}(a_1,\ldots,a_p;\ b_1,\ldots,b_q,\ \alpha;\ -\underline{\underline{h}})\ |\underline{\underline{h}}|^{\alpha-\gamma} \qquad H(2.2)$$

$$= \frac{\Gamma_m(\alpha)}{2\pi i} \int_{\mathrm{Re}\ \underline{\underline{Z}} > \underline{\underline{0}}} e^{\mathrm{tr}\ \underline{\underline{h}}\ \underline{\underline{Z}}}\ {}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;-\underline{\underline{Z}}^{-1})|\underline{\underline{Z}}|^{-\alpha} d\underline{\underline{Z}}$$

Integration in the latter integral is over all symmetric $\underline{\underline{Y}}$ where $\underline{\underline{Z}} = \underline{\underline{X}} + i\,\underline{\underline{Y}}$. The expressions in H(2.1) and H(2.2) are defined for all negative definite matrices and in addition are complex analytic in some region of the space of all complex symmetric matrices (See Herz [4] section 2).

The relation between Herz' inductive definition and that of Constantine given above is easily established using the following theorems due to Constantine:

<u>Theorem C1</u>: Let $\underline{\underline{V}}$ be complex symmetric with positive definite real part and $\underline{\underline{T}}$ be arbitrary complex symmetric. Then provided Re $\alpha > \frac{1}{2}(m-1)$,

$$\int_{\underline{\underline{h}} > \underline{\underline{0}}} e^{-\mathrm{tr}\,\underline{\underline{h}}\,\underline{\underline{V}}} \, |\underline{\underline{h}}|^{\alpha-\gamma} \, C_\kappa(\underline{\underline{h}}\,\underline{\underline{T}}) = \Gamma_m(\alpha,\ \kappa)\ |\underline{\underline{V}}|^{-\alpha}\, C_\kappa(\underline{\underline{V}}^{-1}\,\underline{\underline{T}})$$

<u>Theorem C2</u>: The Laplace transform of $|\underline{\underline{h}}|^{\alpha-\gamma}\, C_\kappa(\underline{\underline{h}})$ is for Re $\alpha > \gamma$,

$$\int_{\underline{\underline{h}} > \underline{\underline{0}}} e^{-\mathrm{tr}\,\underline{\underline{h}}\,\underline{\underline{V}}} \, |\underline{\underline{h}}|^{\alpha-\gamma} \, C_\kappa(\underline{\underline{h}})\ d\underline{\underline{h}} = \Gamma_m(\alpha,\ \kappa)\ |\underline{\underline{V}}|^{-\alpha}\, C_\kappa(\underline{\underline{V}}^{-1})$$

and the corresponding inverse transform is

$$\frac{2^{\frac{1}{2}m(m-1)}}{(2\pi i)^{\frac{1}{2}m(m+1)}} \int_{\mathrm{Re}\,\underline{\underline{V}} > \underline{\underline{0}}} e^{\mathrm{tr}\,\underline{\underline{V}}\,\underline{\underline{h}}}\ |\underline{\underline{V}}|^{-\alpha}\, C_\kappa(\underline{\underline{V}}^{-1})\ d\underline{\underline{V}}$$

$$= \frac{1}{\Gamma_m(\alpha,\ \kappa)}\ |\underline{\underline{h}}|^{\alpha-\gamma}\, C_\kappa(\underline{\underline{h}}) \qquad ,$$

with integration being over $\underline{\underline{V}} = \underline{\underline{X}} + i\ \underline{\underline{Y}}$ with $\underline{\underline{X}} > \underline{\underline{0}}$ and fixed and $\underline{\underline{Y}}$ ranging over all real symmetric matrices in the latter integral.

Theorems C1 and C2 play a central role in the theory and will be used repeatedly. In addition, we shall have frequent need for a lemma of Herz and so state it here:

Lemma H(1.4): Let $M_{k,m}$ be the space of all k x m matrices, $k \geq m$, and $\underline{\underline{Q}}^*$ be an element of the Stieffel manifold $V_{k,m}$; i.e. the collection of all m-tuples of orthonormal k-vectors. Then corresponding to the decomposition of almost all $\underline{\underline{T}} \in M_{k,m}$ into $\underline{\underline{T}} = \underline{\underline{Q}}^* \underline{\underline{R}}^{\frac{1}{2}}$ with $\underline{\underline{R}} > \underline{\underline{0}}$ and $\underline{\underline{Q}}^* \in V_{k,m}$ we have $J(\underline{\underline{T}} \rightarrow (\underline{\underline{Q}}^*, \underline{\underline{R}})) = 2^{-m}\ |\underline{\underline{R}}|^{\frac{1}{2}(k-m-1)}$.

3. Summary

For a convenient overview of the main identities proven here, we summarize them below. The first is an analogue of the main theorem of Constantine [3] quoted as Theorem C1 here. That is, Theorem C1 gives us the Laplace transform of $C_\kappa(\underline{\underline{h}}\ \underline{\underline{T}})\ |\underline{\underline{h}}|^{\alpha-\gamma}$, while Theorem 1 of this paper gives the Laplace transform of $C_\kappa(\underline{\underline{h}}^{-1}\ \underline{\underline{T}})|\underline{\underline{h}}|^{\alpha-\gamma}$. Besides enabling us to derive the Stieltjes transforms of rather complicated functions such as $C_\kappa(-\ \underline{\underline{C}}^t\ \underline{\underline{h}}[\underline{\underline{h}}+\underline{\underline{\psi}}]^{-1}\ \underline{\underline{h}}\ \underline{\underline{C}})\ |\underline{\underline{h}}|^{a-1}\ |\underline{\underline{h}}+\underline{\underline{\psi}}|^{-\alpha}$ (see I(6)), I(1) may be used to find moments of the inverted multivariate beta density.

Identity I(2) is a slight generalization of Theorem C1 that plays an intermediate role in one or two of the proofs while identities I(4b), I(5), and I(6) play an important direct role in the Bayesian analysis of (1.1). However, I(2) is also useful in generating further extensions of natural conjugate families of the type dealt with in [1]. We remark that I(5) stands

in somewhat curious relation to C(6) inasmuch as averaging over O(m) does not split the integrand into a product of zonal polynomials.

I(1) For $\underline{\underline{R}}$ complex symmetric with positive real part and $\underline{\underline{T}}$ arbitrary complex symmetric, provided $\alpha > k_1 + \gamma - 1$,

$$\int_{\underline{\underline{h}} > \underline{\underline{0}}} e^{-\mathrm{tr}\, \underline{\underline{R}}\, \underline{\underline{h}}} |\underline{\underline{h}}|^{\alpha-\gamma} C_\kappa(\underline{\underline{h}}^{-1} \underline{\underline{T}})\, d\underline{\underline{h}}$$

$$= \delta_m(\alpha, \kappa)\, C_\kappa(\underline{\underline{R}}\, \underline{\underline{T}})\, |\underline{\underline{R}}|^{k_m-\alpha} \prod_{j=1}^{m-1} |\underline{\underline{R}}^{[j]}|^{k_j-k_{j+1}}$$

where

$$\delta_m(\alpha, \kappa) = \pi^{\frac{1}{4}m(m-1)} \prod_{j=1}^{m} \Gamma(\alpha-k_j-\tfrac{1}{2}(m-i)) = \pi^{\frac{1}{4}m(m-1)} \prod_{j=1}^{m} \Gamma(\alpha-\gamma-k_j+\tfrac{1}{2}(j+1)) \;.$$

I(2) For $\underline{\underline{Z}}$ complex symmetric with positive real part and $\underline{\underline{T}}$ arbitrary complex symmetric, provided $\alpha > -(k_1+1)$,

$$\int_{\underline{\underline{h}} > \underline{\underline{0}}} C_\kappa(\underline{\underline{h}}\, \underline{\underline{T}})\, e^{-\mathrm{tr}\, \underline{\underline{h}}\, \underline{\underline{Z}}} |\underline{\underline{h}}|^{\alpha+k_m-\gamma} |\underline{\underline{h}}^{[1]}|^{k_1-k_2} \cdots |\underline{\underline{h}}^{[m-1]}|^{k_{m-1}-k_m}\, d\underline{\underline{h}}$$

$$= \theta_m(\alpha, \kappa)\, C_\kappa(\underline{\underline{Z}}^{-1} \underline{\underline{T}})\, |\underline{\underline{Z}}|^{-\alpha-k_1} \prod_{j=2}^{m} |\underline{\underline{Z}}_{[j]}|^{k_j-k_{j+1}}$$

where

$$\theta_m(\alpha, \kappa) = \pi^{\frac{1}{4}m(m-1)} \prod_{j=1}^{m} \Gamma(\alpha+k_j+\tfrac{1}{2}(j+1)) \quad .$$

I(3) Let $\kappa_\delta = (\delta_1,\ldots,\delta_m)$, $\kappa = (k_1,\ldots,k_m)$, $k_i' = k_i + \delta_i$, and $\underline{\underline{T}}$ be arbitrary complex symmetric. Then for $\underline{\underline{V}} > 0$, $\beta > k_1 + \gamma - 1$, and $\alpha > \beta - (k_1'-\delta_m+1)$,

$$\int_{\underline{\underline{h}} > \underline{\underline{0}}} C_\kappa(\underline{\underline{h}}^{-1}\ \underline{\underline{T}})\ \frac{|\underline{\underline{h}}|^{\beta-\gamma}}{|\underline{\underline{h}}+\underline{\underline{V}}|^{(\alpha+\delta_1-\delta_m)}}\ \prod_{j=2}^{m} |(\underline{\underline{h}}+\underline{\underline{V}})_{[j]}|^{\delta_j-\delta_{j+1}}\ d\underline{\underline{h}}$$

$$= \frac{\delta_m(\alpha,\ \kappa)\ \theta_m(\alpha-\beta,\ \kappa)}{\theta_m(\alpha,\ \kappa_\delta)}\ C_\kappa(\underline{\underline{V}}^{-1}\ \underline{\underline{T}})\ |\underline{\underline{V}}|^{-(\alpha-\beta+k_1-\delta_m)m}\ \prod_{j=2}^{m} |\underline{\underline{V}}_{[j]}|^{k_j'-k_{j+1}'}\ .$$

I(4b) For $\underline{\underline{\psi}},\ \underline{\underline{G}},\ \underline{\underline{H}} > \underline{\underline{0}}, \underline{\underline{Z}}^t \in M_{m,m}$ and $\alpha > 0$,

$$\int_{M_{m,m}} e^{-\mathrm{tr}\ \underline{\underline{\psi}}\ \underline{\underline{B}}\ \underline{\underline{B}}^t -\ \mathrm{tr}\ \underline{\underline{H}}\ \underline{\underline{B}}\ \underline{\underline{G}}\ \underline{\underline{B}}^t -\ \mathrm{tr}\ 2\ \underline{\underline{Z}}^t\ \underline{\underline{B}}}\ |\underline{\underline{B}}\ \underline{\underline{B}}^t|^\alpha\ d\underline{\underline{B}}$$

$$= 2^{-m}\ |\underline{\underline{\psi}}|^{-(\alpha+\frac{1}{2}m)}\ \Gamma_m(\alpha+\tfrac{1}{2}m)\ {}_1F_1\ (\alpha+\tfrac{1}{2}m;\ \tfrac{1}{2}m;\ -\underline{\underline{\eta}}^{-1}\ \underline{\underline{Z}}^t\ \underline{\underline{\xi}}^{-1}\ \underline{\underline{Z}})$$

where $\underline{\underline{\eta}}$ and $\underline{\underline{\xi}}$ are diagonal matrices with diagonal elements η_j and ξ_i such that $\xi_i\eta_j = \lambda_i d_j + 1$. Here the λ_i and d_j are the characteristic roots of $\underline{\underline{\psi}}^{-\frac{1}{2}}\ \underline{\underline{H}}\ \underline{\underline{\psi}}^{-\frac{1}{2}}$ and $\underline{\underline{G}}$ respectively.

I(5) For $\underline{\underline{Z}} \in M_{m,m},\ \underline{\underline{R}},\ \underline{\underline{G}},\ \underline{\underline{H}} > \underline{\underline{0}}$ and $\underline{\underline{\xi}}$ and $\underline{\underline{\eta}}$ as defined above,

$$\int_{O(m)} C_\kappa(-2\ \underline{\underline{R}}^{\frac{1}{2}}\ \underline{\underline{Z}}^t \underline{\underline{Q}}\ -\ \underline{\underline{H}}\underline{\underline{Q}}\ \underline{\underline{R}}^{\frac{1}{2}}\ \underline{\underline{G}}\ \underline{\underline{R}}^{\frac{1}{2}}\ \underline{\underline{Q}}^t)\ d\underline{\underline{Q}} = \frac{1}{(\frac{1}{2}m)_\kappa}\ C_\kappa(-\underline{\underline{R}}\ \underline{\underline{\eta}}^{-\frac{1}{2}}\underline{\underline{Z}}^t\underline{\underline{\xi}}^{-1}\ \underline{\underline{Z}}\ \underline{\underline{\eta}}^{-\frac{1}{2}})\quad .$$

I(6) For $\underline{\underline{C}}^t \in M_{m,m},\ \underline{\underline{\psi}},\ \underline{\underline{V}} > \underline{\underline{0}},\ \alpha > 0$, and $a > \gamma$,

$$\int_{\underline{\underline{h}} > \underline{\underline{0}}} e^{-\mathrm{tr}\ \underline{\underline{h}}\ \underline{\underline{V}}}\ \frac{|\underline{\underline{h}}|^{a-\frac{1}{2}}}{|\underline{\underline{h}}+\underline{\underline{\psi}}|^{\alpha+\frac{1}{2}m}}\ C_\kappa(-\ \underline{\underline{C}}^t\ \underline{\underline{h}}[\underline{\underline{h}}+\underline{\underline{\psi}}]^{-1}\ \underline{\underline{h}}\ \underline{\underline{C}})\ d\underline{\underline{h}}$$

$$= 2^m\ \Gamma_m(a+\tfrac{1}{2}m,\ \kappa)\ |\underline{\underline{V}}|^{-(a+\frac{1}{2}m)}\ |\underline{\underline{\psi}}|^{-(\alpha+\frac{1}{2}m)} C_\kappa(-\ \underline{\underline{C}}^t \underline{\underline{V}}^{-1}\ \underline{\underline{\psi}}^{-\frac{1}{2}}[\underline{\underline{\Lambda}}+\underline{\underline{I}}]^{-1}\ \underline{\underline{\psi}}^{-\frac{1}{2}}\underline{\underline{V}}^{-1}\ \underline{\underline{C}})$$

where $\underline{\underline{\Lambda}}$ is the matrix of characteristic roots of $\underline{\underline{\psi}}^{-\frac{1}{2}}\ \underline{\underline{V}}^{-1}\ \underline{\underline{\psi}}^{-\frac{1}{2}}$.

4. Proofs of I(1), I(2), and I(3)

We now prove the analogue of Theorem C1 mentioned in the previous section.

Theorem 1: Let $\underline{\underline{R}}$ be complex symmetric with positive real part and let $\underline{\underline{T}}$ be an arbitrary complex matrix. Then provided $\alpha > k_1 + \gamma - 1$,

$$\int_{\underline{\underline{h}} > \underline{\underline{0}}} e^{-\mathrm{tr}\, \underline{\underline{R}}\, \underline{\underline{h}}} \, |\underline{\underline{h}}|^{\alpha-\gamma} \, C_\kappa(\underline{\underline{h}}^{-1} \underline{\underline{T}}) \, d\underline{\underline{h}} \tag{I(1)}$$

$$= \delta_m(\alpha, \kappa) \, C_\kappa(\underline{\underline{R}}\, \underline{\underline{T}}) \, |\underline{\underline{R}}|^{k_m-\alpha} \prod_{j=1}^{m-1} |\underline{\underline{R}}^{[j]}|^{k_j-k_{j+1}} \quad .$$

Proof: We follow the general outline of the proof of Theorem C1.

First set $\underline{\underline{R}}=\underline{\underline{I}}$. Define the integral with $\underline{\underline{R}}=\underline{\underline{I}}$ as $g(\underline{\underline{T}})$. The function $g(\underline{\underline{T}})$ is clearly symmetric; i.e. $g(\underline{\underline{T}}) = g(\underline{\underline{Q}}\, \underline{\underline{T}}\, \underline{\underline{Q}}^t), \underline{\underline{Q}} \in O(m)$. Then transform from $\underline{\underline{h}}$ to $\underline{\underline{Q}}\, \underline{\underline{h}}\, \underline{\underline{Q}}^t$ and integrate over $O(m)$ using (2.2) to give $g(\underline{\underline{T}}) = [g(\underline{\underline{I}})/C_\kappa(\underline{\underline{I}})]\, C_\kappa(\underline{\underline{T}})$. To evaluate $g(\underline{\underline{I}})/C_\kappa(\underline{\underline{I}}) = \delta_m(\alpha, \kappa)$ let $\underline{\underline{T}}$ be diagonal. Now if we expand $g(\underline{\underline{T}}) = \delta_m(\alpha, \kappa)\, C_\kappa(\underline{\underline{T}})$ using C(11), we find that the coefficient of the first term in the expansion is $d_{\kappa,\kappa}\, \delta_m(\alpha, \kappa)$. Notice, however, that if we expand $C_\kappa(\underline{\underline{h}}^{-1} \underline{\underline{T}})$ using C(11) and then integrate over $\underline{\underline{h}} > \underline{\underline{0}}$ that the first term is

$$d_{\kappa,\kappa}\, \delta_m(\alpha, \kappa) = d_{\kappa,\kappa} \int_{\underline{\underline{h}} > \underline{\underline{0}}} e^{-\mathrm{tr}\, \underline{\underline{h}}} \, |\underline{\underline{h}}|^{\alpha-\gamma} \, |(\underline{\underline{h}}^{-1})^{[1]}|^{k_1-k_2} \, |(\underline{\underline{h}}^{-1})^{[2]}|^{k_2-k_3} \cdots$$

$$|(\underline{\underline{h}}^{-1})^{[m]}|^{k_m} \, d\underline{\underline{h}} \quad .$$

To evaluate $\delta_m(\alpha, \kappa)$ define $\underline{\underline{V}} = \underline{\underline{h}}^{-1}$ and partition $\underline{\underline{V}}$ and $\underline{\underline{h}}$ into

$$\underline{\underline{V}} = \begin{bmatrix} \underline{\underline{V}}^{[j]} & \underline{\underline{V}}_{12}^{[j]} \\ \underline{\underline{V}}_{21}^{[j]} & \underline{\underline{V}}_{[j+1]} \end{bmatrix}, \qquad \underline{\underline{h}} = \begin{bmatrix} \underline{\underline{h}}^{[j]} & \underline{\underline{h}}_{12}^{[j]} \\ \underline{\underline{h}}_{21}^{[j]} & \underline{\underline{h}}_{[j+1]} \end{bmatrix}, \qquad 1 \leq j \leq m-1 \quad .$$

Then $\underline{\underline{V}}^{[j]} = (\underline{\underline{h}}^{[j]} - \underline{\underline{h}}_{12}^{[j]}\, \underline{\underline{h}}_{[j+1]}^{-1}\, \underline{\underline{h}}_{21}^{[j]})^{-1}$, and by a well-known determinental identity, $|\underline{\underline{h}}| = |\underline{\underline{h}}_{[j+1]}|\ |\underline{\underline{h}}^{[j]} - \underline{\underline{h}}_{12}^{[j]}\, \underline{\underline{h}}_{[j+1]}^{-1}\, \underline{\underline{h}}_{21}^{[j]}|$, so that for $j=1,2,\ldots,m-1$, $|\underline{\underline{V}}^{[j]}| = |\underline{\underline{h}}_{[j+1]}|\ |\underline{\underline{h}}|^{-1}$. Consequently the product of determinants on the RHS of (4.1) may be written as

$$|\underline{\underline{h}}|^{\alpha-k_1-\gamma}\ |\underline{\underline{h}}_{[2]}|^{k_1-k_2}\ |\underline{\underline{h}}_{[3]}|^{k_2-k_3} \cdots |\underline{\underline{h}}_{[m]}|^{k_{m-1}-k_m} \quad .$$

This allows us to write

$$\delta_m(\alpha, \kappa) = \int_{\underline{\underline{h}} > \underline{\underline{0}}} e^{-\operatorname{tr} \underline{\underline{h}}}\ |\underline{\underline{h}}|^{\alpha-k_1-\gamma} \prod_{j=2}^{m} |\underline{\underline{h}}_{[j]}|^{k_{j-1}-k_j}\, d\underline{\underline{h}} \quad .$$

Using a simple method devised by Olkin [7] to prove an identity of Bellman [2], we may directly evaluate $\delta_m(\alpha, \kappa)$.

Define an upper triangular matrix $\underline{\underline{L}}$ with $t_{ii} > 0$ such that $\underline{\underline{h}} = \underline{\underline{L}}\, \underline{\underline{L}}^t$. Partition $\underline{\underline{h}}$ and $\underline{\underline{L}}$ as

$$\begin{bmatrix} \underline{\underline{h}}^{[j]} & \underline{\underline{h}}_{12}^{[j]} \\ \underline{\underline{h}}_{21}^{[j]} & \underline{\underline{h}}_{[j+1]} \end{bmatrix}, \qquad \begin{bmatrix} \underline{\underline{L}}_1 & \underline{\underline{L}}_2 \\ \underline{\underline{0}} & \underline{\underline{L}}_3 \end{bmatrix}, \qquad \underline{\underline{h}}_{[j+1]} \text{ and } \underline{\underline{L}}_3\ (m-j) \times (m-j) \quad .$$

Then

$$\underline{\underline{h}}_{[j+1]} = \underline{\underline{L}}_3\, \underline{\underline{L}}_3^t \quad \text{and} \quad |\underline{\underline{h}}_{[j+1]}| = \prod_{i=j+1}^{m} \ell_{ii}^2 \quad .$$

As $J(\underline{\underline{h}} \to \underline{\underline{L}}) = 2^m \prod_{i=1}^{m} \ell_{ii}^{i}$,

$$\delta_m(\alpha, \kappa) = 2^m \int e^{-\Sigma \ell_{ij}^2} \prod_{i=1}^{m} \ell^{2\gamma_m + i - 2\gamma} \prod_{2}^{m} \ell_{ii}^{2\gamma_{m-2}} \cdots \prod_{m-1}^{m} \ell_{ii}^{2\gamma_2} \prod_{m}^{m} \ell_{ii}^{2\gamma_1} \, d\underline{\underline{L}}$$

where $\gamma_m = \alpha - k_1$ and $\gamma_{m-j+1} = k_{j-1} - k_j$, $j=2,\ldots,m$. This is easily verified to equal

$$\pi^{\frac{1}{4}m(m-1)} \prod_{i=1}^{m} \Gamma(\alpha - k_i - \tfrac{1}{2}(m-i)) = \pi^{\frac{1}{4}m(m-1)} \prod_{j=1}^{m} \Gamma(\alpha - \gamma - k_i + \tfrac{1}{2}(j+1)) \quad .$$

Provided $\alpha > k_1 + \frac{1}{2}(m-1)$ the integral exists since $k_{j-1} \geq k_j$, $j=2,\ldots,m$.

To evaluate $g(\underline{\underline{T}})$ when $\underline{\underline{R}} \neq \underline{\underline{I}}$, define $\underline{\underline{R}} = \underline{\underline{U}}\,\underline{\underline{U}}^t$ where $\underline{\underline{U}}$ is lower triangular; i.e. $u_{ij} = 0$ if $i < j$. Transform from $\underline{\underline{h}}$ to $\underline{\underline{U}}^t\,\underline{\underline{h}}\,\underline{\underline{U}} = \underline{\underline{W}}$ in (4.1) and note that $J(\underline{\underline{h}} \to \underline{\underline{W}}) = |\underline{\underline{U}}|^{-(m+1)} = |\underline{\underline{R}}|^{-\frac{1}{2}(m+1)}$, and

$$|\underline{\underline{h}}_{[j]}| = |\underline{\underline{W}}_{[j]}| \prod_{i=j}^{m} u_{ii}^{-2} = |\underline{\underline{W}}_{[j]}| \, |\underline{\underline{R}}^{[j-1]}| \, |\underline{\underline{R}}|^{-1} \quad .$$

Doing the appropriate substitutions yields I(1).

<u>Corollary 1</u>: Let $\underline{\underline{Z}}$ be complex symmetric with positive real part and let $\underline{\underline{T}}$ be an arbitrary complex symmetric matrix. Then if $\alpha > -(k_1+1)$,

$$\int_{\underline{\underline{h}} > \underline{\underline{0}}} C_\kappa(\underline{\underline{h}}\,\underline{\underline{T}}) \, e^{-\mathrm{tr}\,\underline{\underline{h}}\,\underline{\underline{Z}}} \, |\underline{\underline{h}}|^{\alpha-\gamma} \, |\underline{\underline{h}}^{[1]}|^{k_1-k_2} \cdots |\underline{\underline{h}}^{[m]}|^{k_m} \, d\underline{\underline{h}}$$

$$= \theta(\alpha, \kappa) = \pi^{\frac{1}{4}m(m-1)} \prod_{j=1}^{m} \Gamma(\alpha + \tfrac{1}{2}(j+1) + k_j) \quad . \qquad \mathrm{I}(2)$$

Proof: Follow the pattern of the proof of Theorem 1. Set $\underline{\underline{Z}}=\underline{\underline{I}}$ and define the integral above as $h(\underline{\underline{T}})$. Then transforming from $\underline{\underline{h}}$ to $\underline{\underline{Q}}\,\underline{\underline{h}}\,\underline{\underline{Q}}^t$, $\underline{\underline{Q}} \in O(m)$ and integrating over $O(m)$, $h(\underline{\underline{T}}) = [h(\underline{\underline{I}})/C_\kappa(\underline{\underline{I}})]\, C_\kappa(\underline{\underline{T}})$. To evaluate $\theta_m(\alpha, \kappa) = [h(\underline{\underline{I}})/C_\kappa(\underline{\underline{I}})]$, let $\underline{\underline{T}}$ be diagonal, expand both $C_\kappa(\underline{\underline{T}})$ and $C_\kappa(\underline{\underline{h}}\,\underline{\underline{T}})$, then integrate the first term of the latter expansion over $\underline{\underline{h}} > \underline{\underline{0}}$, and match terms.

We find that

$$\theta_m(\alpha, \kappa) = \int_{\underline{\underline{h}} > \underline{\underline{0}}} e^{-\mathrm{tr}\,\underline{\underline{h}}}\, |\underline{\underline{h}}|^{\alpha-\gamma}\, |\underline{\underline{h}}^{[1]}|^{k_1-k_2} \cdots |\underline{\underline{h}}^{[m]}|^{k_m}\, d\underline{\underline{h}} \tag{4.2}$$

and evaluate it as follows: let $\underline{\underline{\tau}}$ be a lower triangular matrix with $\tau_{ii} > 0$ and transform from $\underline{\underline{h}}$ to $\underline{\underline{\tau}}$. This has $J(h \to \tau) = 2^m \prod_{i=1}^m \tau_{ii}^{m-i+1}$, so

$$\theta_m(\alpha, \kappa) = 2^m \int e^{-\Sigma \tau_{ij}^2} \prod_{i=1}^m \tau_{ii}^{2\alpha-i} \prod_1^1 \tau_{ii}^{2(k_1-k_2)} \prod_1^2 \tau_{ii}^{2(k_2-k_3)}$$

$$\cdots \prod_1^{m-1} \tau_{ii}^{2(k_{m-1}-k_m)} \prod_1^m \tau_{ii}^{2k_m}\, d\underline{\underline{\tau}}$$

$$= \pi^{\frac{1}{4}m(m-1)} \prod_{j=1}^m \Gamma(\alpha + k_j + \tfrac{1}{2}(j+1)) .$$

Then we evaluate the integral with $\underline{\underline{Z}} \neq \underline{\underline{I}}$ by transforming in (4.2) from $\underline{\underline{h}}$ to $\underline{\underline{Y}} = \underline{\zeta}^t\, \underline{\underline{R}}\, \underline{\zeta}$ where $\underline{\zeta}$ is upper triangular and $\underline{\zeta}\,\underline{\zeta}^t = \underline{\underline{Z}}$. This has $J(\underline{\underline{h}} \to \underline{\underline{Y}}) = |\underline{\zeta}|^{-(m+1)}$ $|\underline{\underline{Z}}|^{-\frac{1}{2}(m+1)}$ and

$$|\underline{\underline{h}}^{[j]}| = |\underline{\underline{Y}}^{[j]}| \prod_{i=1}^j \zeta_{ii}^{-2} = |\underline{\underline{Y}}^{[j]}|\, |\underline{\underline{Z}}_{[j+1]}|\, |\underline{\underline{Z}}|^{-1} .$$

Substituting gives I(2).

By putting Theorem 1 and Corollary 1 together we obtain an integral identity that may be regarded as a Stieljes transform of $C_\kappa(\underline{\underline{h}}^{-1}\underline{\underline{T}})\,|\underline{\underline{V}}+\underline{\underline{h}}|^{-(\alpha-\gamma)}$.

<u>Theorem 2</u>: Provided $\beta > k_1 + \gamma - 1$, $\alpha > \beta - k_1 - 1$, $\underline{\underline{V}}$ is complex symmetric with positive definite real part, and $\underline{\underline{T}}$ is arbitrary complex symmetric,

$$\int_{\underline{\underline{h}} > \underline{\underline{0}}} C_\kappa(\underline{\underline{h}}^{-1}\,\underline{\underline{T}})\,\frac{|\underline{\underline{h}}|^{\beta-\gamma}}{|\underline{\underline{V}}+\underline{\underline{h}}|^{\alpha}}\,d\underline{\underline{h}} \tag{I(3)}$$

$$= \frac{\delta_m(\beta,\,\kappa)\,\theta_m(\alpha-\beta,\,\kappa)}{\Gamma_m(\alpha)}\;C_\kappa(\underline{\underline{V}}^{-1}\,\underline{\underline{T}})\,|\underline{\underline{V}}|^{-(\alpha-\beta+k_1)}\prod_{j=2}^{m}|\underline{\underline{V}}_{[j]}|^{k_j-k_{j+1}}\,.$$

<u>Proof</u>: Consider the iterated Laplace transform

$$\int_{\underline{\underline{h}} > \underline{\underline{0}}} e^{-\mathrm{tr}\,\underline{\underline{R}}\,\underline{\underline{h}}}\,|\underline{\underline{h}}|^{\beta-\gamma}\,C_\kappa(\underline{\underline{h}}^{-1}\,\underline{\underline{T}})\int_{\underline{\underline{R}} > \underline{\underline{0}}} e^{-\mathrm{tr}\,\underline{\underline{R}}\,\underline{\underline{V}}}\,|\underline{\underline{R}}|^{\alpha-\gamma}\,d\underline{\underline{h}}\,d\underline{\underline{R}}\quad .$$

Integrating first with respect to $\underline{\underline{h}}$, this is by Theorem 1,

$$\delta_m(\beta,\,\kappa)\int_{\underline{\underline{R}} > \underline{\underline{0}}} C_\kappa(\underline{\underline{R}}\,\underline{\underline{T}})\,e^{-\mathrm{tr}\,\underline{\underline{R}}\,\underline{\underline{V}}}\,|\underline{\underline{R}}|^{\alpha-\beta+k_m-\gamma}\prod_{j=1}^{m-1}|\underline{\underline{R}}^{[j]}|^{k_j-k_{j+1}}\,d\underline{\underline{R}}\quad .$$

Now integrating with respect to $\underline{\underline{R}}$ using Corollary 1, the integral becomes

$$\delta_m(\beta,\,\kappa)\,\theta_m(\alpha-\beta,\,\kappa)\,C_\kappa(\underline{\underline{V}}^{-1}\,\underline{\underline{T}})\,|\underline{\underline{V}}|^{-\alpha+\beta-k_1}\prod_{j=2}^{m}|\underline{\underline{V}}_{[j]}|^{k_j-k_{j+1}}\quad . \tag{4.3}$$

On the other hand, integrating first with respect to $\underline{\underline{R}}$ gives

$$\Gamma_m(\alpha)\int_{\underline{\underline{h}} > \underline{\underline{0}}} C_\kappa(\underline{\underline{h}}^{-1}\,\underline{\underline{T}})\,\frac{|\underline{\underline{h}}|^{\beta-\gamma}}{|\underline{\underline{h}}+\underline{\underline{V}}|^{\alpha}}\,d\underline{\underline{h}}\quad . \tag{4.4}$$

Matching (4.3) and (4.4) proves the theorem.

Theorem 2 admits of an easy generalization: in place of the iterated Laplace transform used in the proof of Corollary 1, consider

$$\int_{\underline{\underline{h}} > \underline{\underline{0}}} e^{-\mathrm{tr}\, \underline{\underline{R}}\, \underline{\underline{h}}} \, |\underline{\underline{h}}|^{\beta-\gamma} \, C_\kappa(\underline{\underline{h}}^{-1}\, \underline{\underline{T}}) \int_{\underline{\underline{R}} > \underline{\underline{0}}} e^{-\mathrm{tr}\, \underline{\underline{R}}\, \underline{\underline{V}}} \, |\underline{\underline{R}}|^{\alpha-\gamma} \prod_{j=1}^{m-1} |\underline{\underline{R}}^{[j]}|^{\delta_j - \delta_{j+1}} \, d\underline{\underline{h}}\, d\underline{\underline{R}} \tag{4.5}$$

for $\delta_j - \delta_{j+1} \geq 0$. Integrating first with respect to $\underline{\underline{h}}$, the integral is

$$\delta_m(\beta,\ \kappa) \int_{\underline{\underline{R}} > \underline{\underline{0}}} C_\kappa(\underline{\underline{R}}\ \underline{\underline{T}})\ e^{-\mathrm{tr}\ \underline{\underline{R}}\ \underline{\underline{V}}}\ |\underline{\underline{R}}|^{\alpha-\beta+k_m-\gamma} \prod_{j=1}^{m-1} |\underline{\underline{R}}^{[j]}|^{k'_j - k'_{j+1}}\ d\underline{\underline{R}}$$

where $k'_j = k_j + \delta_j$, and by Corollary 1 this equals

$$\delta(\beta;\ \kappa)\ \theta(\alpha-\beta,\ \kappa)\ C_\kappa(\underline{\underline{V}}^{-1}\ \underline{\underline{T}})\ |\underline{\underline{V}}|^{-\alpha+\beta-k_1+\delta_m} \prod_{j=2}^{m} |\underline{\underline{V}}_{[j]}|^{k'_j - k'_{j+1}}\ .$$

However, if we integrate (4.5) with respect to $\underline{\underline{R}}$ first, the integral is by Corollary 1,

$$\theta(\alpha,\ \kappa_\delta) \int_{\underline{\underline{h}} > \underline{\underline{0}}} C_\kappa(\underline{\underline{h}}^{-1}\ \underline{\underline{T}})\ \frac{|\underline{\underline{h}}|^{\beta-\gamma}}{|\underline{\underline{h}}+\underline{\underline{V}}|^{-(\alpha+\delta_1-\delta_m)}} \prod_{j=2}^{m} |(\underline{\underline{h}}+\underline{\underline{V}})_{[j]}|^{\delta_j-\delta_{j+1}}\ d\underline{\underline{h}}$$

where κ_δ is the partition $(\delta_1,\ldots,\delta_m)$. And so we obtain

<u>Corollary 2</u>: Let $\kappa_\delta = (\delta_1,\ldots,\delta_m)$, $\kappa = (k_1,\ldots,k_m)$, $k'_i = k_i + \delta_i$, and $\underline{\underline{T}}$ be arbitrary complex symmetric. Then for $\underline{\underline{V}} > \underline{\underline{0}}$, $\beta > k_1 + \gamma - 1$, and $\alpha > \beta - (k'_1 - \delta_m + 1)$,

$$\int_{\underline{\underline{h}} > \underline{\underline{0}}} C_\kappa(\underline{\underline{h}}^{-1}\ \underline{\underline{T}})\ \frac{|\underline{\underline{h}}|^{\beta-\gamma}}{|\underline{\underline{h}}+\underline{\underline{V}}|^{(\alpha+\delta_1-\delta_m)}} \prod_{j=2}^{m} |(\underline{\underline{h}}+\underline{\underline{V}})_{[j]}|^{\delta_j-\delta_{j+1}}\ d\underline{\underline{h}}$$

$$= \frac{\delta_m(\beta,\ \kappa)\ \theta_m(\alpha-\beta,\ \kappa)}{\theta_m(\alpha,\ \kappa_\delta)}\ C_\kappa(\underline{\underline{V}}^{-1}\ \underline{\underline{T}})\ |\underline{\underline{V}}|^{-(\alpha-\beta+k_1-\delta_m)} \prod_{j=2}^{m} |\underline{\underline{V}}_{[j]}|^{k'_j-k'_{j+1}}\ .$$

This is I(3).

5. Characteristic Function of Inverted Multivariate Beta Density

Identity I(1) gives us an easy method for finding moments of the inverted multivariate beta density. First we derive a formal expression for its characteristic function in terms of ${}_2F_0$ and then derive first and second moments using I(1).[†]

We say that the random (m x m) matrix $\tilde{\underline{\underline{U}}}$ has inverted multivariate beta density $iB(\underline{\underline{I}}, \alpha, \beta)$ if

$$iB(\underline{\underline{I}}, \alpha, \beta) = B_m^{-1}(\alpha, \beta) \frac{|\underline{\underline{U}}|^{\alpha-\gamma}}{|\underline{\underline{I}}+\underline{\underline{U}}|^{\alpha+\beta}} \quad , \qquad \alpha, \beta > \tfrac{1}{2}\gamma - 1 \qquad ,$$

where $B_m(\alpha, \beta) \equiv \dfrac{\Gamma_m(\alpha)\ \Gamma_m(\beta)}{\Gamma_m(\alpha+\beta)}$. The characteristic function of $i\underline{\underline{B}}(\underline{\underline{I}}, \alpha, \beta)$ is, for $\underline{\underline{A}}$ complex symmetric and Re $\underline{\underline{A}} = \underline{\underline{0}}$,

$$[B_m(\alpha, \beta)]^{-1} \int_{\underline{\underline{U}} > \underline{\underline{0}}} \frac{e^{-\operatorname{tr} \underline{\underline{A}}\, \underline{\underline{U}}}\ |\underline{\underline{U}}|^{\alpha-\gamma}}{|\underline{\underline{I}}+\underline{\underline{U}}|^{\alpha+\beta}}\, d\underline{\underline{U}} = \frac{\Gamma_m(\alpha+\beta)}{\Gamma_m(\beta)}\ {}_2F_0(\alpha+\beta, \alpha; -\underline{\underline{A}}^{-1})\ |\underline{\underline{A}}|^{-\alpha} . \tag{5.1}$$

<u>Proof</u>: Define for Re $\underline{\underline{A}} = \underline{\underline{0}}$ and $\alpha, \beta > \gamma - 1$,

$$I_B(\underline{\underline{A}}, \alpha, \beta) = B_m^{-1}(\alpha, \beta) \int_{\underline{\underline{U}} > \underline{\underline{0}}} \frac{e^{-\operatorname{tr} \underline{\underline{A}}\, \underline{\underline{U}}}\ |\underline{\underline{U}}|^{\alpha-\gamma}}{|\underline{\underline{I}}+\underline{\underline{U}}|^{\alpha+\beta}}\, d\underline{\underline{U}} \quad .$$

Then as ${}_1F_0(\alpha+\beta; -\underline{\underline{U}}) = |\underline{\underline{I}}+\underline{\underline{U}}|^{-(\alpha+\beta)}$,

---

[†]For another way of deriving first and second moments of $iB(\underline{\underline{I}}, \alpha, \beta)$ see Martin [7].

$$I_B(\underline{\underline{A}}, \alpha, \beta) = B_m^{-1}(\alpha, \beta) \int_{\underline{\underline{U}} > \underline{\underline{0}}} e^{-\mathrm{tr}\, \underline{\underline{A}}\, \underline{\underline{U}}} |\underline{\underline{U}}|^{\alpha-\gamma} {}_1F_0(\alpha+\beta; -\underline{\underline{U}})\, d\underline{\underline{U}}$$

and this equals (5.1) by virtue of definition H(2.1).

Expression (5.1) for the characteristic function of $\tilde{\underline{\underline{U}}}$ is not in a form convenient for computing moments of $\tilde{\underline{\underline{U}}}$. Theorem 1, however, gives us an easy way of finding them. Since the characteristic function

$$E(e^{-\mathrm{tr}\, \underline{\underline{A}}\, \tilde{\underline{\underline{U}}}}) = \sum_{k=0}^{\infty} \frac{1}{k!} E(-\mathrm{tr}\, \underline{\underline{A}}\, \tilde{\underline{\underline{U}}})^k = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{1}{k!} E(C_\kappa(-\underline{\underline{A}}\, \tilde{\underline{\underline{U}}})) \quad ,$$

matching coefficients of appropriate powers of elements of $\underline{\underline{A}}$ in the two expansions will give the moments.

Write (5.1) as an iterated Laplace transform

$$\frac{1}{\Gamma_m(\alpha)\, \Gamma_m(\beta)} \int_{\underline{\underline{U}} > \underline{\underline{0}}} {}_0F_0(-\underline{\underline{A}}\, \underline{\underline{U}})\, e^{-\mathrm{tr}\, \underline{\underline{U}}\, \underline{\underline{Z}}} |\underline{\underline{U}}|^{\alpha-\gamma} \cdot \int_{\underline{\underline{Z}} > \underline{\underline{0}}} e^{-\mathrm{tr}\, \underline{\underline{Z}}} |\underline{\underline{Z}}|^{\alpha+\beta-\gamma}\, d\underline{\underline{U}}\, d\underline{\underline{Z}} \quad . \tag{5.2}$$

Then by definition H(2.1), (5.2) is

$$\frac{1}{\Gamma_m(\beta)} \int_{\underline{\underline{Z}} > \underline{\underline{0}}} {}_1F_0(\alpha; -\underline{\underline{A}}\, \underline{\underline{Z}}^{-1})\, e^{-\mathrm{tr}\, \underline{\underline{Z}}} |\underline{\underline{Z}}|^{\beta-\gamma}\, d\underline{\underline{Z}} \tag{5.3}$$

A generic term in the expansion of (5.3) in zonal polynomials is

$$\frac{(\alpha)_\kappa}{k!\, \Gamma_m(\beta)} \int_{\underline{\underline{Z}} > \underline{\underline{0}}} C_\kappa(-\underline{\underline{A}}\, \underline{\underline{Z}}^{-1})\, e^{-\mathrm{tr}\, \underline{\underline{Z}}} |\underline{\underline{Z}}|^{\beta-\gamma}\, d\underline{\underline{Z}}$$

and by Theorem 1, provided $\beta > k_1 - \gamma - 1$, this term is

$$E(C_\kappa(-\underline{\underline{A}}\ \tilde{\underline{\underline{U}}})) = \frac{\delta_m(\beta,\ \kappa)\ (\alpha)_\kappa}{k!\ \Gamma_m(\beta)}\ C_\kappa(-\underline{\underline{A}}) \qquad .$$

A little algebra shows that

$$\frac{\delta_m(\beta,\ (1))\ (\alpha)_{(1)}}{\Gamma_m(\beta)} = \frac{\alpha}{\beta-\gamma}\ , \qquad \frac{\delta_m(\beta,\ (2,0))\ (\alpha)_{(2,0)}}{\Gamma_m(\beta)} = \frac{\alpha(\alpha+1)}{(\beta-\gamma)\ (\beta-\gamma-1)}\ ,$$

and

$$\frac{\delta_m(\beta,\ (1,1))\ \alpha_{(1,1)}}{\Gamma_m(\beta)} = \frac{\alpha(\alpha-\frac{1}{2})}{(\beta-\gamma)\ (\beta-\gamma+\frac{1}{2})}\ ,$$

so as

$$E(-\mathrm{tr}\ \underline{\underline{A}}\ \tilde{\underline{\underline{U}}}) = E\ C_{(1)}(-\underline{\underline{A}}\ \tilde{\underline{\underline{U}}})$$

we have immediately

$$E(\tilde{\underline{\underline{U}}}) = \frac{\alpha}{\beta-\gamma}\ \underline{\underline{I}} \qquad . \tag{5.4}$$

To find variances and covariances of the $\tilde{u}_{ij}$s observe that

$$\begin{aligned} E(-\mathrm{tr}\ \underline{\underline{A}}\ \tilde{\underline{\underline{U}}})^2 &= E\ C_{(2,0)}(-\underline{\underline{A}}\ \tilde{\underline{\underline{U}}}) + E\ C_{(1,1)}(-\underline{\underline{A}}\ \tilde{\underline{\underline{U}}}) \\ &= \frac{\alpha(\alpha+1)}{(\beta-\gamma)\ (\beta-\gamma-1)}\ C_{(2,0)}(-\underline{\underline{A}}) + \frac{\alpha(\alpha-\frac{1}{2})}{(\beta-\gamma)\ (\beta-\gamma+\frac{1}{2})}\ C_{(1,1)}(-\underline{\underline{A}}) \quad . \end{aligned} \tag{5.5}$$

Expanding $C_{(2,0)}(-\underline{\underline{A}})$ and $C_{(1,1)}(-\underline{\underline{A}})$ in terms of the elementary symmetric functions of the latent roots of $-\underline{\underline{A}}$ gives (See James [5] and Martin [7])

$$C_{(2,0)}(-\underline{\underline{A}}) = \sum_{i=1}^{m} a_{ii}^2 + \frac{1}{3}\sum_{i=1}^{m}\sum_{j\neq i}(a_{ii}\ a_{jj} + 2a_{ij}^2) \tag{5.6}$$

and

$$C_{(1,1)}(-\underline{\underline{A}}) = \frac{2}{3}\sum_{i=1}^{m}\sum_{j\neq i}(a_{ii}\ a_{jj} - a_{ij}^2) \quad . \tag{5.7}$$

Substituting (5.6) and (5.7) in (5.5) and then computing $E(\tilde{u}_{ij}\ \tilde{u}_{\ell p})$ $1 \leq i, j, \ell, p \leq m$ by matching terms on both sides of (5.5) yields

$$\begin{aligned}
\operatorname{Var}(\tilde{u}_{ii}) &= \frac{\alpha(\alpha+\beta-\gamma)}{(\beta-\gamma)^2\ (\beta-\gamma-1)}\ , \\
\operatorname{Var}(\tilde{u}_{ij}) &= \frac{\alpha(\alpha+1)}{3(\beta-\gamma)\ (\beta-\gamma-1)}\ , \qquad i \neq j\ , \qquad (5.8) \\
\operatorname{Cov}(\tilde{u}_{ii}, \tilde{u}_{jj}) &= \frac{\alpha[3\alpha-(\beta-\gamma)\ (2\alpha-1)]}{3(\beta-\gamma)^2\ (\beta-\gamma-1)}\ , \qquad i \neq j\ ,
\end{aligned}$$

and all other covariances are 0. Martin [7] obtains (5.8) in a quite different way.

6. Some Additional Laplace Transforms

The identities of this section were motivated by a desire to find explicit expressions for several densities that arise in a Bayesian analysis of the system (1.1). In the next section we show how the identities I(4), I(5), and I(6) apply when a prior density of the form $f(\underline{\underline{B}}) \propto |\underline{\underline{B}}\ \underline{\underline{B}}^t|^\alpha \exp - \underline{\underline{H}}[\underline{\underline{B}}-\bar{\underline{\underline{B}}}]\ \underline{\underline{G}}[\underline{\underline{B}}-\bar{\underline{\underline{B}}}]^t$ is assigned to $\tilde{\underline{\underline{B}}}$ with range set $M_{m,m}$ and a Normal-Wishart (natural conjugate) prior is assigned to $\tilde{\underline{\underline{\Gamma}}}$, $\tilde{\underline{\underline{h}}}|\underline{\underline{B}}$.

We first define the Laplace transform L_g of $g(\underline{\underline{B}})$ with domain $M_{m,m}$ as $\int_{M_{m,m}} e^{-\operatorname{tr} 2\, \underline{\underline{Z}}^t\, \underline{\underline{B}}}\ g(\underline{\underline{B}})\, d\underline{\underline{B}}$, $\underline{\underline{Z}} \in M_{m,m}$, and prove

<u>Lemma 6.1</u>: Let $g(B) = e^{-\operatorname{tr} \underline{\underline{H}}\ \underline{\underline{B}}\ \underline{\underline{G}}\ \underline{\underline{B}}^t}\ |\underline{\underline{B}}\ \underline{\underline{B}}^t|^\alpha$ and $h(\underline{\underline{B}}) = e^{-\operatorname{tr} \underline{\underline{\psi}}\ \underline{\underline{B}}\ \underline{\underline{B}}^t}\ g(\underline{\underline{B}})$; $\underline{\underline{\psi}}, \underline{\underline{G}}, \underline{\underline{H}} > \underline{\underline{0}}$ and $\alpha > 0$. Then

$$L_g = 2^{-m}\ \Gamma_m(\alpha+\tfrac{1}{2}m)\ |\underline{\underline{G}}\ \underline{\underline{H}}|^{-(\alpha+\frac{1}{2}m)}\ {}_1F_1\ (\alpha+\tfrac{1}{2}m;\ \tfrac{1}{2}m;\ -\ \underline{\underline{G}}^{-1}\ \underline{\underline{Z}}^t\ \underline{\underline{H}}^{-1}\ \underline{\underline{Z}}) \qquad I(4)$$

and

$$L^\circ h = 2^{-m}\,\Gamma_m(\alpha+\tfrac{1}{2}m)\;|\underset{=}{\Psi}|^{-(\alpha+\frac{1}{2}m)}\;{}_1F_1\,(\alpha+\tfrac{1}{2}m;\ \tfrac{1}{2}m;\ -\ \underset{=}{\eta}^{-1}\,\underset{=}{Z}^t\,\underset{=}{\xi}^{-1}\,\underset{=}{Z}) \qquad \text{I(4b)}$$

where $\underset{=}{\eta}$ and $\underset{=}{\xi}$ are diagonal matrices defined by (6.5) below.

Proof: Transform from $\underset{=}{B}$ to $\underset{=}{W} = \underset{=}{H}^{\frac{1}{2}}\,\underset{=}{B}\,\underset{=}{G}^{\frac{1}{2}}$. This transform has $J(\underset{=}{B}\to\underset{=}{W}) = |\underset{=}{H}|^{-\frac{1}{2}m}\,|\underset{=}{G}|^{-\frac{1}{2}m}$. Then transform from $\underset{=}{W}$ to $\underset{=}{U} = \underset{=}{Q}\,\underset{=}{R}^{\frac{1}{2}}$, $\underset{=}{Q}\in O(m)$, $\underset{=}{R} > \underset{=}{0}$. By Lemma H(1.4), $J(\underset{=}{W}\to\underset{=}{U}) = 2^{-m}\,|\underset{=}{R}|^{-\frac{1}{2}}$ so that $L^\circ g$ is

$$2^{-m}\,|\underset{=}{G}\,\underset{=}{H}|^{-(\alpha+\frac{1}{2}m)}\int_{\underset{=}{R}>\underset{=}{0}} e^{-\mathrm{tr}\,\underset{=}{R}}\,|\underset{=}{R}|^{\alpha-\frac{1}{2}}\int_{O(m)} e^{-\mathrm{tr}\,2[\underset{=}{R}^{\frac{1}{2}}\,\underset{=}{G}^{-\frac{1}{2}}\,\underset{=}{Z}^t\,\underset{=}{H}^{-\frac{1}{2}}]\underset{=}{Q}}\,d\underset{=}{Q}\,d\underset{=}{R}. \qquad (6.1)$$

The inner integral is absolutely convergent and possesses an expansion in zonal polynomials (J(27) and definition H(2.1)) that allows us to write it as

$$2^{-m}\,|\underset{=}{G}\,\underset{=}{H}|^{-(\alpha+\frac{1}{2}m)}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{k!\,(\frac{1}{2}m)_\kappa}\int_{\underset{=}{R}>\underset{=}{0}} e^{-\mathrm{tr}\,\underset{=}{R}}\,|\underset{=}{R}|^{\alpha-\frac{1}{2}}\,C_\kappa(-\ \underset{=}{R}\,\underset{=}{G}^{-\frac{1}{2}}\underset{=}{Z}^t\underset{=}{H}^{-1}\underset{=}{Z}\,\underset{=}{G}^{-\frac{1}{2}})\,d\underset{=}{R} \qquad (6.2)$$

By Theorem C1,

$$\begin{aligned}\int_{\underset{=}{R}>\underset{=}{0}} e^{-\mathrm{tr}\,\underset{=}{R}}\,|\underset{=}{R}|^{\alpha-\frac{1}{2}}\,C_\kappa(-\ \underset{=}{R}\,\underset{=}{G}^{-\frac{1}{2}}\,\underset{=}{Z}^t\,\underset{=}{H}^{-1}\,\underset{=}{Z}\,\underset{=}{G}^{-\frac{1}{2}})&\\ = \Gamma_m(\alpha+\tfrac{1}{2}m,\ \kappa)\,C_\kappa(-\ \underset{=}{G}^{-1}\,\underset{=}{Z}^t\,\underset{=}{H}^{-1}\,\underset{=}{Z})&\end{aligned} \qquad (6.3)$$

and substitution of (6.3) in (6.2) gives $L^\circ g$.

To evaluate $L^\circ h$, define $\underset{=}{W} = \underset{=}{Q}\,\underset{=}{\Psi}^{\frac{1}{2}}\,\underset{=}{B}\,\underset{=}{\theta}$ where $\underset{=}{Q}$, $\underset{=}{\theta}\in O(m)$ and

$$\begin{aligned}\underset{=}{Q}^t\,\underset{=}{\Delta}\,\underset{=}{Q} &= \underset{=}{\Psi}^{-\frac{1}{2}}\,\underset{=}{H}\,\underset{=}{\Psi}^{-\frac{1}{2}}, &&\underset{=}{\Delta}\ \text{diagonal},\\ \underset{=}{\theta}\,\underset{=}{D}\,\underset{=}{\theta}^t &= \underset{=}{G}, &&\underset{=}{D}\ \text{diagonal}.\end{aligned} \qquad (6.4)$$

Then $J(\underline{\underline{B}} \to \underline{\underline{W}}) = |\underline{\underline{\psi}}|^{-\frac{1}{2}m}$ and

$$\text{tr } 2\,\underline{\underline{Z}}^t\,\underline{\underline{B}} + \text{tr } \underline{\underline{H}}\,\underline{\underline{B}}\,\underline{\underline{G}}\,\underline{\underline{B}}^t + \underline{\underline{\psi}}\,\underline{\underline{B}}\,\underline{\underline{B}}^t = \text{tr } 2[\underline{\underline{\theta}}^t\underline{\underline{Z}}^t\underline{\underline{\psi}}^{-\frac{1}{2}}\underline{\underline{Q}}^t]\underline{\underline{W}} + \text{tr } \underline{\underline{D}}\,\underline{\underline{W}}^t\underline{\underline{\Lambda}}\,\underline{\underline{W}} + \text{tr } \underline{\underline{W}}^t\underline{\underline{W}} .$$

As $\underline{\underline{D}}$ and $\underline{\underline{\Lambda}}$ are diagonal

$$\text{tr } \underline{\underline{D}}\,\underline{\underline{W}}^t\underline{\underline{\Lambda}}\,\underline{\underline{W}} + \text{tr } \underline{\underline{W}}^t\underline{\underline{W}} = \sum_{i,j} \lambda_i d_j w_{ij}^2 + \sum_{i,j} w_{ij}^2 = \sum_{i,j} (\lambda_i d_j + 1) w_{ij}^2 .$$

Define ξ_i and η_j, $1 \le i, j \le m$, so that $\xi_i\eta_j = \lambda_i d_j + 1$ and then write

$$\text{tr } \underline{\underline{D}}\,\underline{\underline{W}}^t\,\underline{\underline{\Lambda}}\,\underline{\underline{W}} + \text{tr } \underline{\underline{W}}^t\,\underline{\underline{W}} = \text{tr } \underline{\underline{\xi}}\,\underline{\underline{W}}\,\underline{\underline{\eta}}\,\underline{\underline{W}}^t \quad , \tag{6.5}$$

where $\underline{\underline{\xi}}$ and $\underline{\underline{\eta}}$ are (m x m) diagonal matrices with diagonal entries ξ_i and η_j respectively. Substituting the RHS of (6.5) in the integral after transforming from $\underline{\underline{B}}$ to $\underline{\underline{W}}$ and then applying Log yields Loh.

We now use Lemma 6.1 to prove Theorem 3 (I(5)) which is also needed in the analysis of the system (1.1).

<u>Theorem 3</u>: For $\underline{\underline{Z}} \in M_{m,m}$, $\underline{\underline{R}}$, $\underline{\underline{H}}$, $\underline{\underline{H}} > \underline{\underline{0}}$, and $\underline{\underline{\xi}}$ and $\underline{\underline{\eta}}$ as defined in (6.5)

$$\int_{O(m)} C_\kappa(-2\,\underline{\underline{R}}^{\frac{1}{2}}\,\underline{\underline{Z}}^t\,\underline{\underline{Q}} - \underline{\underline{H}}\,\underline{\underline{Q}}\,\underline{\underline{R}}^{\frac{1}{2}}\,\underline{\underline{G}}\,\underline{\underline{R}}^{\frac{1}{2}}\,\underline{\underline{Q}}^t)\, d\underline{\underline{Q}} = \frac{1}{(\frac{1}{2}m)_\kappa} C_\kappa(-\underline{\underline{R}}\,\underline{\underline{\eta}}^{-\frac{1}{2}}\underline{\underline{Z}}^t\underline{\underline{\xi}}^{-1}\underline{\underline{Z}}\,\underline{\underline{\eta}}^{-\frac{1}{2}}) \tag{I(5)}$$

<u>Proof</u>: Without loss of generality, set $\underline{\underline{\psi}}=\underline{\underline{I}}$ in the integrand of Loh, expand $e^{-\text{tr } 2\,\underline{\underline{Z}}^t\,\underline{\underline{B}} - \text{tr } \underline{\underline{H}}\,\underline{\underline{B}}\,\underline{\underline{G}}\,\underline{\underline{B}}^t}$ in zonal polynomials, and transform from $\underline{\underline{B}}$ to $(\underline{\underline{Q}}, \underline{\underline{R}})$ with $\underline{\underline{B}} = \underline{\underline{Q}}\,\underline{\underline{R}}^{\frac{1}{2}}$, $\underline{\underline{Q}} \in O(m)$, $\underline{\underline{R}} > \underline{\underline{0}}$. Then Loh may be written as

$$2^{-m} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{1}{k!} \int_{O(m)} \int_{\underline{\underline{R}} > \underline{\underline{0}}} e^{-\text{tr } \underline{\underline{R}}} |\underline{\underline{R}}|^{\alpha-\frac{1}{2}} \tag{6.6}$$
$$C_\kappa(-2\,\underline{\underline{Z}}^t\,\underline{\underline{Q}}\,\underline{\underline{R}}^{\frac{1}{2}} - \underline{\underline{H}}\,\underline{\underline{Q}}\,\underline{\underline{R}}^{\frac{1}{2}}\,\underline{\underline{G}}\,\underline{\underline{R}}^{\frac{1}{2}}\,\underline{\underline{Q}}^t)\, d\underline{\underline{Q}}\, d\underline{\underline{R}} \quad .$$

Matching (6.6) with L°h as shown in I(4b) of this section,

$$\int_{O(m)} \int_{R>0} e^{-\mathrm{tr}\, R} \, |R|^{\alpha-\frac{1}{2}} \, C_\kappa(-2\, R^{\frac{1}{2}} Z^t Q - H\, Q\, R^{\frac{1}{2}} G\, R^{\frac{1}{2}} Q^t) \, dQ \, dR$$

$$= \frac{\Gamma_m(\alpha+\frac{1}{2}m,\ \kappa)}{(\frac{1}{2}m)_\kappa} \; C_\kappa(-\ \eta^{-1} Z^t \xi^{-1} Z) \qquad (6.7)$$

where ξ and η are as defined in (6.4). Now define the function

$$\Phi(R|Z,\ G,\ H) = \int_{O(m)} C_\kappa(-2\, R^{\frac{1}{2}} Z^t Q - H\, Q\, R^{\frac{1}{2}} G\, R^{\frac{1}{2}} Q^t) \, dQ \quad .$$

We may write the LHS of (6.7) as

$$\int_{R>0} e^{-\mathrm{tr}\, R} \, |R|^{\alpha-\frac{1}{2}} \, \Phi(R|Z,\ G,\ H) \, dR \quad . \qquad (6.8)$$

Examine Theorem C1 with $V=I$ and $T = -\ \eta^{-\frac{1}{2}} Z^t \xi^{-1} Z\, \eta^{-\frac{1}{2}}$. The theorem then states that for $\beta > \gamma - 1$, T complex symmetric,

$$\int_{R>0} e^{-\mathrm{tr}\, R} \, |R|^{\beta-\gamma} \, C_\kappa(R\, T) \, dR = \Gamma_m(\beta,\ \kappa) \; C_\kappa(T) \quad . \qquad (6.9)$$

If we set $\beta = \alpha+\frac{1}{2}m$ and compare the RHS of (6.9) with (6.7), by the uniqueness of the Laplace transform, $\Phi(R|Z,\ G,\ H) = \frac{1}{(\frac{1}{2}m)_\kappa} \; C_\kappa(R\ T)$, and this proves the theorem.

Theorem 3 is the key to I(6) and we prove the latter by evaluating

$$\int_{h>0} \int_{M_{m,m}} e^{-\mathrm{tr}\, h\, V} \, |h|^{a-\frac{1}{2}} \, e^{-\mathrm{tr}\, 2\, h\, B\, C^t - \mathrm{tr}[h+\psi] B\, B^t} \, |B\, B^t|^{\alpha} \, dB \, dh \qquad (6.10)$$

$$V,\ \psi > 0,\ \alpha,\ a > 0 \quad ,$$

two ways. Integrating first over $M_{m,m}$ using Lemma 6.1 gives the integral as

$$2^{-m}\,\Gamma_m(\alpha+\tfrac{1}{2}m)\int_{\underline{\underline{h}}>\underline{\underline{0}}} e^{-\mathrm{tr}\,\underline{\underline{h}}\,\underline{\underline{V}}}\,\frac{|\underline{\underline{h}}|^{a-\frac{1}{2}}}{|\underline{\underline{h}}+\underline{\underline{\psi}}|^{\alpha+\frac{1}{2}m}}\;{}_1F_1(\alpha+\tfrac{1}{2}m;\,\tfrac{1}{2}m;\,-\,\underline{\underline{C}}^t\underline{\underline{h}}[\underline{\underline{h}}+\underline{\underline{\psi}}]^{-1}\underline{\underline{h}}\,\underline{\underline{C}})\,d\underline{\underline{h}}\,. \tag{6.11}$$

Now integrate (6.10) over $\underline{\underline{h}} > \underline{\underline{0}}$ using the definition H(2.1):

$$e^{-\mathrm{tr}\,\underline{\underline{\psi}}\,\underline{\underline{B}}\,\underline{\underline{B}}^t}\,|\underline{\underline{B}}\,\underline{\underline{B}}^t|^{\alpha}\int_{\underline{\underline{h}}>\underline{\underline{0}}} e^{-\mathrm{tr}\,\underline{\underline{h}}\,\underline{\underline{V}}}\,|\underline{\underline{h}}|^{a-\frac{1}{2}}\;{}_0F_0(\underline{\underline{h}}[-2\,\underline{\underline{B}}\,\underline{\underline{C}}^t-\underline{\underline{B}}\,\underline{\underline{B}}^t])\,d\underline{\underline{h}}$$

$$=\Gamma_m(a+\tfrac{1}{2}m)\;|\underline{\underline{V}}|^{-(a+\frac{1}{2}m)}\;e^{-\mathrm{tr}\,\underline{\underline{\psi}}\,\underline{\underline{B}}\,\underline{\underline{B}}^t}|\underline{\underline{B}}\,\underline{\underline{B}}^t|^{\alpha}\;{}_1F_0(a+\tfrac{1}{2}m;\,\underline{\underline{V}}^{-1}[-2\underline{\underline{B}}\,\underline{\underline{C}}^t-\underline{\underline{B}}\,\underline{\underline{B}}^t])\,. \tag{6.12}$$

Transforming from $\underline{\underline{B}}$ to $\underline{\underline{W}} = \underline{\underline{\psi}}^{\frac{1}{2}}\,\underline{\underline{B}}$ with $J(\underline{\underline{B}}\to\underline{\underline{W}}) = |\underline{\underline{\psi}}|^{\frac{1}{2}m}$ and then to $(\underline{\underline{Q}},\,\underline{\underline{R}})$ with $\underline{\underline{W}} = \underline{\underline{Q}}\,\underline{\underline{R}}^{\frac{1}{2}}$, $\underline{\underline{Q}}\in O(m)$, $\underline{\underline{R}}>\underline{\underline{0}}$, we may write the RHS of (6.12) as

$$\Gamma_m(a+\tfrac{1}{2}m)\;|\underline{\underline{V}}|^{-(a+\frac{1}{2}m)}\;|\underline{\underline{\psi}}|^{-(\alpha+\frac{1}{2}m)} \tag{6.13}$$

$$\int_{R>0}\int_{O(m)} e^{-\mathrm{tr}\,\underline{\underline{R}}}\,|\underline{\underline{R}}|^{\alpha-\frac{1}{2}}\;{}_1F_0(a+\tfrac{1}{2}m;\,-2\,\underline{\underline{R}}^{\frac{1}{2}}\underline{\underline{Z}}^t\underline{\underline{Q}}-\underline{\underline{H}}\,\underline{\underline{Q}}\,\underline{\underline{R}}^{\frac{1}{2}}\underline{\underline{G}}\,\underline{\underline{R}}^{\frac{1}{2}}\underline{\underline{Q}}^t)\,d\underline{\underline{Q}}\,d\underline{\underline{R}}$$

where $\underline{\underline{Z}}^t = \underline{\underline{C}}^t\,\underline{\underline{V}}^{-1}\,\underline{\underline{\psi}}^{-\frac{1}{2}}$, $\underline{\underline{H}} = \underline{\underline{\psi}}^{-\frac{1}{2}}\underline{\underline{V}}^{-1}\,\underline{\underline{\psi}}^{-\frac{1}{2}}$, and $\underline{\underline{G}}=\underline{\underline{I}}$. By expanding ${}_1F_0$ in the integrand immediately above in zonal polynomials using definition J(10) and applying Theorem 3 we may write this integral as

$$\Gamma_m(a+\tfrac{1}{2}m)\;\Gamma_m(\alpha+\tfrac{1}{2}m)\;|\underline{\underline{V}}|^{-(a+\frac{1}{2}m)}\;|\underline{\underline{\psi}}|^{-(\alpha+\frac{1}{2}m)} \tag{6.14}$$

$$\cdot\,{}_2F_1(a+\tfrac{1}{2}m,\,\alpha+\tfrac{1}{2}m;\,\tfrac{1}{2}m;\,\underline{\underline{Z}}^t(\underline{\underline{\Lambda}}+\underline{\underline{I}})^{-1}\,\underline{\underline{Z}})$$

where $\underset{=}{\Lambda}$ is the matrix of characteristic roots of $\underset{=}{\psi}^{-\frac{1}{2}}\ \underset{=}{V}^{-1}\ \underset{=}{\psi}^{-\frac{1}{2}}$ and $\underset{=}{Z}^{t} = \underset{=}{C}^{t}\ \underset{=}{V}^{-1}\ \underset{=}{\psi}^{-\frac{1}{2}}$.
Matching (6.11) and (6.14) gives

$$\int_{\underset{=}{h} > \underset{=}{0}} e^{-\mathrm{tr}\ \underset{=}{h}\ \underset{=}{V}} \frac{|\underset{=}{h}|^{a-\frac{1}{2}}}{|\underset{=}{h}+\underset{=}{\psi}|^{\alpha+\frac{1}{2}m}}\ {}_1F_1(\alpha+\tfrac{1}{2}m;\ \tfrac{1}{2}m;\ -\ \underset{=}{C}^{t}\ \underset{=}{h}[\underset{=}{h}+\underset{=}{\psi}]^{-1}\ \underset{=}{h}\ \underset{=}{C})\ d\underset{=}{h} \tag{6.15}$$

$$= 2^{m}\ \Gamma_m(a+\tfrac{1}{2}m)\ |\underset{=}{V}|^{-(a+\frac{1}{2}m)}\ |\underset{=}{\psi}|^{-(\alpha+\frac{1}{2}m)}\cdot\ {}_2F_1(a+\tfrac{1}{2}m,\alpha+\tfrac{1}{2}m;\tfrac{1}{2}m;\underset{=}{Z}^{t}(\underset{=}{\Lambda}+\underset{=}{I})^{-1}\underset{=}{Z})$$

Expanding both sides of (6.15) in zonal polynomials and matching terms gives I(6):

$$\int_{\underset{=}{h} > \underset{=}{0}} e^{-\mathrm{tr}\ \underset{=}{h}\ \underset{=}{V}} \frac{|\underset{=}{h}|^{a-\frac{1}{2}}}{|\underset{=}{h}+\underset{=}{\psi}|^{\alpha+\frac{1}{2}m}}\ C_\kappa(-\ \underset{=}{C}^{t}\ \underset{=}{h}[\underset{=}{h}+\underset{=}{\psi}]^{-1}\ \underset{=}{h}\ \underset{=}{C})\ d\underset{=}{h} \tag{I(6)}$$

$$= 2^{m}\ \Gamma_m(a+\tfrac{1}{2}m,\ \kappa)\ |\underset{=}{V}|^{-(a+\frac{1}{2}m)}|\underset{=}{\psi}|^{-(\alpha+\frac{1}{2}m)}\ C_\kappa(-\ \underset{=}{C}^{t}\underset{=}{V}^{-1}\underset{=}{\psi}^{-\frac{1}{2}}[\underset{=}{\Lambda}+\underset{=}{I}]^{-1}\underset{=}{\psi}^{-\frac{1}{2}}\underset{=}{V}^{-1}\underset{=}{C})\ .$$

7. Application to Simultaneous Equation System (1.1)

We now apply the results of the last section to the simultaneous equation system (1.1). Suppose that a (natural conjugate) Normal-Wishart prior with kernel

$$e^{-\mathrm{tr}\ \underset{=}{h}\ \underset{=}{\epsilon}}\ |\underset{=}{h}|^{\frac{1}{2}(\nu-1)} e^{-\frac{1}{2}\mathrm{tr}\ \underset{=}{h}\{[-\underset{=}{\Gamma}-\underset{=}{B}\ \underset{=}{P}]\underset{=}{V}[-\underset{=}{\Gamma}-\underset{=}{B}\ \underset{=}{P}]^{t}\}}\ |\underset{=}{h}|^{\frac{1}{2}r} \tag{7.1}$$

$$\underset{=}{V},\ \underset{=}{\epsilon} > \underset{=}{0},\ \nu > 0\quad ,$$

is assigned to $(\tilde{\underset{=}{\Gamma}}, \tilde{\underset{=}{h}})$ given $\tilde{\underset{=}{B}}=\underset{=}{B}$. If we then assign a prior $f(\underset{=}{B})$ to $\tilde{\underset{=}{B}}$ in order to find the prior of $(\tilde{\underset{=}{\Gamma}}, \tilde{\underset{=}{h}})$ unconditional as regards $\tilde{\underset{=}{B}}$ and the prior of $\tilde{\underset{=}{\Gamma}}$ unconditional as regards $\tilde{\underset{=}{B}}$ and $\tilde{\underset{=}{h}}$, we must first integrate the joint density of $(\tilde{\underset{=}{\Gamma}}, \tilde{\underset{=}{h}}, \tilde{\underset{=}{B}})$ over $M_{m,m}$ and then over $M_{m,m}$ and $\underset{=}{h} > \underset{=}{0}$.

A "natural" class of priors to assign to $\tilde{\mathbf{B}}$ is, as stated in the introduction

$$f(\mathbf{B}) = e^{-\frac{1}{2}\mathrm{tr}\ \boldsymbol{\psi}[\mathbf{B}-\bar{\mathbf{B}}]\mathbf{E}[\mathbf{B}-\bar{\mathbf{B}}]^t} |\mathbf{B}\ \mathbf{B}^t|^{\alpha} \quad , \qquad \boldsymbol{\psi},\ \mathbf{E} > \mathbf{0},\ \bar{\mathbf{B}} \in M_{m,m},\ \alpha > 0 \ .$$

Then the kernel of the density of $(\tilde{\boldsymbol{\Gamma}}, \tilde{\mathbf{h}})$ unconditional as regards $\tilde{\mathbf{B}}$ is proportional to

$$e^{-\frac{1}{2}\mathrm{tr}\ \mathbf{h}\ \boldsymbol{\epsilon}}\ |\mathbf{h}|^{\frac{1}{2}(\nu+r-1)} \int_{M_{m,m}} f(\mathbf{B})\ e^{-\frac{1}{2}\mathrm{tr}\ \mathbf{h}\{[-\boldsymbol{\Gamma}-\mathbf{B}\ \mathbf{P}]\mathbf{V}[-\boldsymbol{\Gamma}-\mathbf{B}\ \mathbf{P}]^t\}}\ d\mathbf{B}\ . \qquad (7.2)$$

Transforming from $\mathbf{B}$ to $\mathbf{W} = \mathbf{B}\ \mathbf{E}^{\frac{1}{2}}$ and defining

$$\mathbf{A} = \mathbf{E}^{-\frac{1}{2}}\ \mathbf{P}\ \mathbf{V} \quad , \quad \mathbf{C} = \mathbf{E}^{\frac{1}{2}}\ \bar{\mathbf{B}}^t\ \boldsymbol{\psi} \quad , \quad \mathbf{G} = \mathbf{E}^{-\frac{1}{2}}\ \mathbf{P}\ \mathbf{V}\ \mathbf{P}^t\ \mathbf{E}^{-\frac{1}{2}}$$

the kernel (7.1) may be written as

$$e^{-\frac{1}{2}\mathrm{tr}\ \mathbf{h}\ \boldsymbol{\epsilon}}\ |\mathbf{h}|^{\frac{1}{2}(\nu+r-1)} \cdot e^{-\frac{1}{2}\mathrm{tr}\ \mathbf{h}\ \boldsymbol{\Gamma}\ \mathbf{V}\ \boldsymbol{\Gamma}^t}$$

$$\cdot \int_{M_{m,m}} e^{-\mathrm{tr}[\mathbf{A}\ \boldsymbol{\Gamma}^t\ \mathbf{h}-\mathbf{C}]\mathbf{W}\ -\ \frac{1}{2}\mathrm{tr}\ \boldsymbol{\psi}\ \mathbf{W}\ \mathbf{W}^t\ -\ \frac{1}{2}\mathrm{tr}\ \mathbf{h}\ \mathbf{W}\ \mathbf{G}\ \mathbf{W}^t}\ |\mathbf{W}\ \mathbf{W}^t|^{\alpha}\ d\mathbf{W} \ .$$

Defining, for $\mathbf{Q} \in O(m)$ and $\boldsymbol{\theta} \in O(m)$,

$$\boldsymbol{\Lambda} = \mathbf{Q}\ \boldsymbol{\psi}^{-\frac{1}{2}}\ \mathbf{h}\ \boldsymbol{\psi}^{-\frac{1}{2}}\ \mathbf{Q}^t \quad , \qquad \boldsymbol{\Lambda} \text{ diagonal} ,$$

$$\mathbf{D} = \boldsymbol{\theta}^t\ \mathbf{G}\ \boldsymbol{\theta} \quad , \qquad \mathbf{D} \text{ diagonal} ,$$

and ξ_i and η_j such that $\xi_i \eta_j = \lambda_i d_j + 1$, $1 \leq i,\ j \leq m$ we find the kernel by using Lemma 6.1:

$$(\tilde{\Gamma}, \tilde{h}) \propto W(\epsilon + \Gamma V \Gamma^t, \nu+r)\ {}_1F_1(\alpha+\tfrac{1}{2}m; \tfrac{1}{2}m; \eta^{-1}[A\ \Gamma^t h-C]\xi^{-1}[A\ \Gamma^t h-C]^t)$$

where $W(\epsilon + \Gamma V \Gamma^t, \nu+r)$ is the Wishart density with parameter $(\epsilon + \Gamma V \Gamma^t, \nu+r)$.

To illustrate further the ideas developed here we now show that the marginal density of $\tilde{\Gamma}$ can be written in terms of ${}_2F_1$ times a determinental factor. In order to keep the algebra simple, we do this for the special case when $f(B)$ is assigned so that $\bar{B}=0$ and $E=I$. Then the kernel of the marginal density of $\tilde{\Gamma}$ is proportional to

$$\int_{h>0}\int_{M_{m,m}} e^{-\frac{1}{2}\mathrm{tr}\ h\{\epsilon + [-\Gamma-B\ P]V[-\Gamma-B\ P]^t\}} |h|^{\frac{1}{2}(\nu+r-1)} \cdot e^{-\mathrm{tr}\ \psi\ B\ B^t} |B\ B^t|^{\alpha}\ dB\ dh \tag{7.3}$$

which is in turn proportional to†

$$\int_{M_{m,m}} e^{-\mathrm{tr}\ \psi\ B\ B^t} |B\ B^t|^{\alpha}\ {}_1F_0(\tfrac{1}{2}(\nu+r); -\tfrac{1}{2}\epsilon^{-1}[-\Gamma-B\ P]V[-\Gamma-B\ P]^t)\ dB \quad .$$

If we transform from B to $W = \psi^{\frac{1}{2}} B$, and expand $-\frac{1}{2}\epsilon^{-1}[-\Gamma-\psi^{-\frac{1}{2}} W\ P]V[-\Gamma-\psi^{-\frac{1}{2}} W\ P]^t$, we may rewrite the argument of ${}_1F_0$ immediately above as $-K^{-1}[2\ \psi^{-\frac{1}{2}} W\ P\ V\ \Gamma^t + \psi^{-\frac{1}{2}} W\ P\ V\ P^t\ W\ \psi^{-\frac{1}{2}}]$ where $K = \frac{1}{2}[\epsilon + \Gamma\ V\ \Gamma^t]$. Setting $Z^t = P\ V\ \Gamma^t\ K^{-1}\ \psi^{-\frac{1}{2}}$, $H=\psi^{-1}$ and $G = P\ V\ P^t$ and then transforming from W to (Q, R) with $W = Q\ R^{\frac{1}{2}}$, $Q \in O(m)$, $R > 0$, the integrand is essentially that of (6.13), so the kernel of the marginal density of $\tilde{\Gamma}$ is, using Theorem 3 as in the development of (6.14),

---

†Using ${}_1F_0(\alpha; -A) = |I+A|^{-\alpha}$.

$$|\underline{\underline{\epsilon}} + \underline{\underline{\Gamma}}\ \underline{\underline{V}}\ \underline{\underline{\Gamma}}^t|^{-\frac{1}{2}(\nu+r+m)}\ {}_2F_1(\tfrac{1}{2}(\nu+r+m),\ \alpha+\tfrac{1}{2}m;\ \tfrac{1}{2}m;\ \underline{\underline{\eta}}^{-1}\ \underline{\underline{Z}}^t\ \underline{\underline{\xi}}^{-1}\ \underline{\underline{Z}})$$

where $\underline{\underline{Z}}$ is as defined immediately above and $\underline{\underline{\eta}}$ and $\underline{\underline{\xi}}$ have definitions paralleling (6.5).

If in place of the prior on $\tilde{\underline{\underline{B}}}$ used above we assign a prior

$$f(\underline{\underline{B}}) = e^{-tr\ \underline{\underline{\psi}}\ \underline{\underline{B}}[\underline{\underline{P}}\ \underline{\underline{V}}\ \underline{\underline{P}}^t]\underline{\underline{B}}}\ |\underline{\underline{B}}\ \underline{\underline{B}}^t|^{\alpha}$$

to $\tilde{\underline{\underline{B}}}$, by transforming from $\underline{\underline{B}}$ to $\underline{\underline{W}} = \underline{\underline{B}}[\underline{\underline{P}}\ \underline{\underline{V}}\ \underline{\underline{P}}^t]^{\frac{1}{2}}$, setting the $\underline{\underline{V}}$ of (6.10) equal to $\frac{1}{2}[\underline{\underline{\epsilon}} + \underline{\underline{\Gamma}}\ \underline{\underline{V}}\ \underline{\underline{\Gamma}}^t]$ and the a of (6.10) equal to $\frac{1}{2}(\nu+m)$, we find that the kernel of the marginal density of $\tilde{\underline{\underline{\Gamma}}}$ is identical to (6.10) with $\underline{\underline{C}}^t = [\underline{\underline{P}}\ \underline{\underline{V}}\ \underline{\underline{P}}^t]^{\frac{1}{2}}\ \underline{\underline{P}}\ \underline{\underline{V}}\ \underline{\underline{\Gamma}}^t$.

It follows from (6.14) that the kernel of the marginal density of $\tilde{\underline{\underline{\Gamma}}}$ is, with this prior,

$$|\underline{\underline{\epsilon}} + \underline{\underline{\Gamma}}\ \underline{\underline{V}}\ \underline{\underline{\Gamma}}^t|^{-(a+\frac{1}{2}m)}\ {}_2F_1(\tfrac{1}{2}(\nu+r+m),\ \alpha+\tfrac{1}{2}m;\ \tfrac{1}{2}m;\ \underline{\underline{Z}}^t[\underline{\underline{\Delta}}+\underline{\underline{I}}]^{-1}\ \underline{\underline{Z}})$$

where $\underline{\underline{Z}}^t = [\underline{\underline{P}}\ \underline{\underline{V}}\ \underline{\underline{P}}^t]^{\frac{1}{2}}\ \underline{\underline{P}}\ \underline{\underline{V}}\ \underline{\underline{\Gamma}}^t[\underline{\underline{\epsilon}} + \underline{\underline{\Gamma}}\ \underline{\underline{V}}\ \underline{\underline{\Gamma}}^t]^{-1}\ \underline{\underline{\psi}}^{-\frac{1}{2}}$ and $\underline{\underline{\Delta}}$ is the matrix of characteristic roots of $\underline{\underline{\psi}}^{-\frac{1}{2}}[\underline{\underline{\epsilon}} + \underline{\underline{\Gamma}}\ \underline{\underline{V}}\ \underline{\underline{\Gamma}}^t]^{-1}\ \underline{\underline{\psi}}^{-\frac{1}{2}}$.

REFERENCES

[1] Ando, A, and Kaufman, G.M. (1964) "Extended Natural Conjugate Distributions for the Multinormal Process," Working Paper No. 80-64, Sloan School of Management, Massachusetts Institute of Technology.

[2] Bellman, R. (1956) "A Generalization of Some Integral Identities Due to Ingham and Siegel," Duke Mathematics Journal, Vol. 23, pp. 571-577.

[3] Constantine, A.G. (1963) "Some Noncentral Distribution Problems in Multivariate Analysis," Annals of Mathematical Statistics, Vol. 34, pp. 1270-1285.

[4] Herz, C.S. (1955) "Bessel Functions of Matrix Argument," Annals of Mathematics, Vol. 61, pp. 474-523.

[5] James, A.T. (1964) "Distributions of Matrix Variates and Latent Roots Derived from Normal Samples," Annals of Mathematical Statistics, Vol. 35, pp. 475-501.

[6] __________. (1960) "The Distribution of the Latent Roots of the Covariance Matrix," Annals of Mathematical Statistics, Vol. 31, pp. 151-158.

[7] Martin, J. (1965) "Some Moment Formulas for Multivariate Analysis," (unpublished).

[8] Olkin, I. (1959) "A Class of Integral Identities with Matrix Argument," Duke Mathematics Journal, Vol. 26, pp. 207-213.

[9] Raiffa, H. and Schlaifer, R. (1961) Applied Statistical Decision Theory, Division of Research, Harvard Business School, Boston, Massachusetts.